Chapter 26
Proofs Answers

NB: If your proof does not have the same number of general variables as these solutions, it is probable there is a loss of generality and the proof will therefore be incomplete. When comparing your own proofs with the exemplars here, compare very carefully to ensure that all steps are rigorously covered.

Exercise 26.1A

Throughout this exercise it is assumed that $k, j \in \mathbb{Z}$.

1. $m = 2k$ and $n = 2j + 1$

Then $m + n = (2k) + (2j + 1)$

$= 2(k + j) + 1$ which is odd.

2. $m = 2k + 1$ and $n = 2j + 1$

Then $m + n = (2k + 1) + (2j + 1)$

$= 2(k + j + 1)$ which is even.

3. $m = 2k$

Then $9m - 4 = 9(2k) - 4$

$= 2(9k - 2)$ which is even.

4. $m = 2k + 1$

Then $m^2 = (2k + 1)^2$

$= 4k^2 + 4k + 1$

$= 2(2k^2 + 2k) + 1$ which is odd.

5. $m = 2k + 1$ and $n = 2j + 1$

Then $mn = (2k + 1)(2j + 1)$

$= 4kj + 2k + 2j + 1$

$= 2(2kj + k + j) + 1$ which is odd.

6. $m = 2k$ and $n = 2j + 1$

Then $mn = (2k)(2j + 1)$

$= 4kj + 2k + 2j + 1$

$= 2(2kj + k + j) + 1$ which is even.

7. $m = 2k + 1$

Then $m^3 = (2k + 1)^3$

$= 8k^3 + 12k^2 + 6k + 1$

$= 2(4k^3 + 6k^2 + 3k) + 1$ which is odd.

8. $m = 2k$

Then $m^2 + 3m = (2k)^2 + 3(2k)$

$= 4k^2 + 6k$

$= 2(k^2 + 3k)$ which is even.

9. $n = 2j + 1$

Then $n^2 + 1 = (2j + 1)^2 + 1$

$= 4j^2 + 4j + 2$

$= 2(j^2 + 2j + 1)$ which is even.

10. $n = 2j + 1$

Then $3n^2 = 3(2j + 1)^2$

$= 12k^2 + 12k + 3$

$= 2(6k^2 + 6k + 1) + 1$ which is odd.

11. $n = 2j + 1$

Then $n - 5 = (2j + 1) - 5$

$= 2(j - 2)$ which is even.

12. $m = 2k + 1$ and $n = 2j + 1$

Then $m^2 + n^2 = (2k + 1)^2 + (2j + 1)^2$

$= 4k^2 + 4k + 1 + 4j^2 + 4j + 1$

$= 2(2k^2 + 2k + 2j^2 + 2j + 1)$ which is even.

13. $m = 2k + 1$ and $n = 2j$

Then $m^3 + n^3 - 1 = (2k + 1)^3 + (2j)^2 - 1$

$= 8k^3 + 12k^2 + 6k + 1 + 4j^2 - 1$

$= 2(4k^3 + 3k^2 + 3k + 2j^2)$ which is even.

14. $n = 2j + 1$

Then $n^3 - n = (2j + 1)^3 - (2j + 1)$

$= 8j^3 + 12j^2 + 6j + 1 - (2j + 1)$

$= 2(4j^3 + 6j^2 + 2j)$ which is even.

15. $n = 2j$

Then $n^2 - 5n = (2j)^3 - 5(2j)$

$= 8j^3 - 10j$

$= 2(4j^3 - 5j)$ which is even.

16. $m = 2k + 1$ and $n = 2j$

Then $2m^2 + 3n = 2(2k + 1)^2 + 3(2j)$

$= 8k^2 + 8k + 2 + 6j$

$= 2(4k^2 + 4k + 1 + 3j)$ which is even.

17. $n = 2j + 1$

Then $n^2 - 1 = (2j + 1)^2 - 1$

$= 4j^2 + 4j$

$= 4(j^2 + j)$ which is divisible by 4.

18. $n = 2j + 1$

Then $n^2 - 1 = (2j + 1)^2 - 1$

$= 4j^2 + 4j$

$= 4j(j + 1)$

$j(j + 1)$ is the product of two consecutive integers and can be represented by $2m$, $m \in \mathbb{Z}$.

$$= 4(2m)$$
$$= 8m \text{ which is divisible by } 8$$

Exercise 26.1B

Throughout this exercise it is assumed that $k, m \in \mathbb{Z}$.

1. $v = ku$ and $w = mv$

   Then $v + w = ku + mv$

   $= ku + m(ku)$

   $= u(k + mk)$ which is divisible by u.

2. $v = 2k$ and $w = 3m$

   Then $vw = (2k)(3m)$

   $= 6(km)$ which is divisible by 6.

3. $b = ka$ and $c = mb$

   Then $c = mb$

   $= m(ka)$

   $= a(mk)$ which is divisible by a.

4. $b = ka$ and $d = mc$

   Then $bd = (ka)(mc)$

   $= (ac)(km)$ which is divisible by ac.

5. $b = ka$

   Then $bc = (ka)c$

   $= a(kc)$ which is divisible by a

   Or

   If $c = ma$

   Then $bc = b(ma)$

   $= a(bm)$ which is divisible by a.

6. $b = ka$

   Then $bx = (ka)x$

   $= a(kx)$ which is divisible by a.

Exercise 26.2

There may be alternative solutions to these counterexamples. Provided is one possible answer.

1. $x = 9$ is between 2 and 11 but is not prime

2. $2 + 4 = 6$ is the sum of a prime number and an even integer, but the answer is not odd

3. $p = 11$, $q = 8$, then $11 - 8 = 3$ which is a prime number

4. $x = 4$, then $4^2 + 4 + 5 = 25$ which is not prime

5. $x = 3$, then $3^4 + 1 = 82$ which is not prime

6. $2 + 3 = 5$ is the sum of two prime numbers, but the answer is not even

7. $x = 2$, then $2 + 2 = 4$ which is not prime

8. $x = 2$, 2 is not divisible by 4, however $2^2 = 4$ which is divisible by 4

9. $x = -1$ since $-4 \not> -3$

10. $x = 0$, since $0 + 0^2 \not> 0$

11. $x = -1$ since $\sqrt{(-1)^2} \neq -1$

12. $x = 3$ since $2 \times 2 \neq 2^3$

13. $x = 1$ since $1^2 \not> 1$

14. $x = -3$ since $(-3 + 3)^2 \not> (-3)^2$

15. $a = 1, b = -1$ since $1^2 \not> (-1)^2$

16. $x = 30, y = 60$ since $\sin(30 + 60) \neq \sin 30 + \sin 60$

17. $x = 1$ since $e^{\ln 0} \neq 0$

18. An equilateral triangle does not have any obtuse angles

19. The lines $y = 2x + 3$ and $y = 2x - 6$ are parallel, not

perpendicular

20. $n = 11$ produces $2047 = 23 \times 89$ as is therefore not prime

Exercise 26.3A

Throughout this exercise it is assumed that $k, j \in \mathbb{Z}$.

1. Contradiction statement: 'Assume mn is odd, then either m or n is even'
$m = 2k$ and $n = 2j + 1$
Then $mn = (2k)(2j + 1)$
$= 4kj + 2 = 2(2kj + 1)$ which is even.
This is a contradiction to the assumption, therefore the original statement must be true.

2. Contradiction statement: 'Assume $m^2 + n^2$ is odd, then both m and n are even'
$m = 2k$ and $n = 2j$
Then $m^2 + n^2 = (2k)^2 + (2j)^2$
$= 4k^2 + 4j^2 = 2(4j^2 + 4j^2)$ which is even.
This is a contradiction to the assumption, therefore the original statement must be true.

3. Contradiction statement: 'Assume mn is even, then both m and n are odd'
$m = 2k + 1$ and $n = 2j + 1$
Then $mn = (2k + 1)(2j + 1)$
$= 4kj + 2k + 2j + 1 = 2(4kj + k + j) + 1$ which is odd.
This is a contradiction to the assumption, therefore the original statement must be true.

4. Contradiction statement: 'Assume $m - n$ is odd, then m and n are both even'
$m = 2k$ and $n = 2j$
Then $m - n = (2k) - (2j)$
$= 2(k - j)$ which is even.
This is a contradiction to the assumption, therefore the original statement must be true.

5. Contradiction statement: 'Assume that if m and n are consecutive integers, then $m + n$ is even'
$m = 2k$ and $n = 2k + 1$
Then $m + n = (2k) = (2k + 1)$
$= 4k + 1 = 2(2k) + 1$ which is odd.
This is a contradiction to the assumption, therefore the original statement must be true.

6. Contradiction statement: 'Assume that if m and n are 2 non-zero consecutive integers then mn is odd'
$m = 2k$ and $n = 2k + 1$, then m, n are consecutive integers
Then $mn = (2k)(2k + 1)$
$= 4k^2 + 2k = 2(2k^2 + k)$ which is even.
This is a contradiction to the assumption, therefore the original statement must be true.

Exercise 26.3B

Throughout this exercise it is assumed that $a, b \in \mathbb{N}$ and $k \in \mathbb{Z}$.

1.

(a) Assume that $\sqrt{2}$ is rational.
Then $\sqrt{2} = \dfrac{a}{b}$ where a and b have no common factors.
$2 = \dfrac{a^2}{b^2}$
$2b^2 = a^2$
This implies that a^2 is a multiple of 2, which means that a is a multiple of $2 \Rightarrow a = 2k$
$2b^2 = (2k)^2$
$2b^2 = 4k^2$
$b^2 = 2k^2$
This implies that b^2 is also a multiple of 2. And in turn, that b is a multiple of 2.
We now have that both a and b are multiples of 2. This contradicts the assumption that a and b have no common factors. Our assumption must be false and the original statement is proven true.

(b) Assume that $\sqrt{5}$ is rational.
Then $\sqrt{5} = \dfrac{a}{b}$ where a and b have no common factors.
$5 = \dfrac{a^2}{b^2}$
$5b^2 = a^2$
This implies that a^2 is a multiple of 5, which means that a is a multiple of $5 \Rightarrow a = 5k$
$5b^2 = (5k)^2$
$5b^2 = 25k^2$
$b^2 = 5k^2$
This implies that b^2 is also a multiple of 5. And in turn, that b is a multiple of 5.

We now have that both a and b are multiples of 5. This contradicts the assumption that a and b have no common factors. Our assumption must be false and the original statement is proven true.

(c) Assume that $\sqrt{7}$ is rational.

Then $\sqrt{7} = \dfrac{a}{b}$ where a and b have no common factors.

$7 = \dfrac{a^2}{b^2}$

$7b^2 = a^2$

This implies that a^2 is a multiple of 7, which means that a is a multiple of $7 \Rightarrow a = 7k$

$7b^2 = (7k)^2$

$7b^2 = 49k^2$

$b^2 = 7k^2$

This implies that b^2 is also a multiple of 7. And in turn, that b is a multiple of 7.

We now have that both a and b are multiples of 7. This contradicts the assumption that a and b have no common factors. Our assumption must be false and the original statement is proven true.

(d) Assume that $\sqrt{11}$ is rational.

Then $\sqrt{11} = \dfrac{a}{b}$ where a and b have no common factors.

$11 = \dfrac{a^2}{b^2}$

$11b^2 = a^2$

This implies that a^2 is a multiple of 11, which means that a is a multiple of $11 \Rightarrow a = 11k$

$11b^2 = (11k)^2$

$11b^2 = 121k^2$

$b^2 = 11k^2$

This implies that b^2 is also a multiple of 11. And in turn, that b is a multiple of 11.

We now have that both a and b are multiples of 11. This contradicts the assumption that a and b have no common factors. Our assumption must be false and the original statement is proven true.

(e) Assume that $\sqrt{13}$ is rational.

Then $\sqrt{13} = \dfrac{a}{b}$ where a and b have no common factors.

$13 = \dfrac{a^2}{b^2}$

$13b^2 = a^2$

This implies that a^2 is a multiple of 13, which means that a is a multiple of $13 \Rightarrow a = 13k$

$13b^2 = (13k)^2$

$13b^2 = 169k^2$

$b^2 = 13k^2$

This implies that b^2 is also a multiple of 13. And in turn,

that b is a multiple of 13.

We now have that both a and b are multiples of 13. This contradicts the assumption that a and b have no common factors. Our assumption must be false and the original statement is proven true.

(f) Assume that $\sqrt{17}$ is rational.

Then $\sqrt{17} = \dfrac{a}{b}$ where a and b have no common factors.

$17 = \dfrac{a^2}{b^2}$

$17b^2 = a^2$

This implies that a^2 is a multiple of 17, which means that a is a multiple of $17 \Rightarrow a = 17k$

$17b^2 = (17k)^2$

$17b^2 = 289k^2$

$b^2 = 17k^2$

This implies that b^2 is also a multiple of 17. And in turn, that b is a multiple of 17.

We now have that both a and b are multiples of 17. This contradicts the assumption that a and b have no common factors. Our assumption must be false and the original statement is proven true.

(g) Assume that $\sqrt{19}$ is rational.

Then $\sqrt{19} = \dfrac{a}{b}$ where a and b have no common factors.

$19 = \dfrac{a^2}{b^2}$

$19b^2 = a^2$

This implies that a^2 is a multiple of 19, which means that a is a multiple of $19 \Rightarrow a = 19k$

$19b^2 = (19k)^2$

$19b^2 = 361k^2$

$b^2 = 19k^2$

This implies that b^2 is also a multiple of 19. And in turn, that b is a multiple of 19.

We now have that both a and b are multiples of 19. This contradicts the assumption that a and b have no common factors. Our assumption must be false and the original statement is proven true.

(h) Assume that $\sqrt{23}$ is rational.

Then $\sqrt{23} = \dfrac{a}{b}$ where a and b have no common factors.

$23 = \dfrac{a^2}{b^2}$

$23b^2 = a^2$

This implies that a^2 is a multiple of 23, which means that a is a multiple of $23 \Rightarrow a = 23k$

$23b^2 = (23k)^2$

$23b^2 = 529k^2$

$b^2 = 23k^2$

This implies that b^2 is also a multiple of 23. And in turn, that b is a multiple of 23.

We now have that both a and b are multiples of 23. This contradicts the assumption that a and b have no common factors. Our assumption must be false and the original statement is proven true.

2.

(a) Assume that $\sqrt[3]{2}$ is rational.

Then $\sqrt[3]{2} = \dfrac{a}{b}$ where a and b have no common factors.

$2 = \dfrac{a^3}{b^3}$

$2b^3 = a^3$

This implies that a^3 is a multiple of 2, which means that a is a multiple of $2 \Rightarrow a = 2k$

$2b^3 = (2k)^3$

$2b^3 = 8k^3$

$b^3 = 4k^3 = 2(2k^3)$

This implies that b^3 is also a multiple of 2. And in turn, that b is a multiple of 2.

We now have that both a and b are multiples of 2. This contradicts the assumption that a and b have no common factors. Our assumption must be false and the original statement is proven true.

(b) Assume that $\sqrt[3]{3}$ is rational.

Then $\sqrt[3]{3} = \dfrac{a}{b}$ where a and b have no common factors.

$3 = \dfrac{a^3}{b^3}$

$3b^3 = a^3$

This implies that a^3 is a multiple of 3, which means that a is a multiple of $3 \Rightarrow a = 3k$

$3b^3 = (3k)^3$

$3b^3 = 27k^3$

$b^3 = 9k^3 = 3(3k^3)$

This implies that b^3 is also a multiple of 3. And in turn, that b is a multiple of 3.

We now have that both a and b are multiples of 3. This contradicts the assumption that a and b have no common factors. Our assumption must be false and the original statement is proven true.

(c) Assume that $\sqrt[3]{5}$ is rational.

Then $\sqrt[3]{5} = \dfrac{a}{b}$ where a and b have no common factors.

$5 = \dfrac{a^3}{b^3}$

$5b^3 = a^3$

This implies that a^3 is a multiple of 5, which means that a is a multiple of $5 \Rightarrow a = 5k$

$5b^3 = (5k)^3$

$5b^3 = 125k^3$

$b^3 = 25k^3 = 5(5k^3)$

This implies that b^3 is also a multiple of 5. And in turn, that b is a multiple of 5.

We now have that both a and b are multiples of 5. This contradicts the assumption that a and b have no common factors. Our assumption must be false and the original statement is proven true.

(d) Assume that $\sqrt[3]{6}$ is rational.

Then $\sqrt[3]{6} = \dfrac{a}{b}$ where a and b have no common factors.

$6 = \dfrac{a^3}{b^3}$

$6b^3 = a^3$

This implies that a^3 is a multiple of 6, which means that a is a multiple of $6 \Rightarrow a = 6k$

$6b^3 = (6k)^3$

$6b^3 = 216k^3$

$b^3 = 36k^3 = 6(6k^3)$

This implies that b^3 is also a multiple of 6. And in turn, that b is a multiple of 6.

We now have that both a and b are multiples of 6. This contradicts the assumption that a and b have no common factors. Our assumption must be false and the original statement is proven true.

(e) Assume that $\sqrt[4]{2}$ is rational.

Then $\sqrt[4]{2} = \dfrac{a}{b}$ where a and b have no common factors.

$2 = \dfrac{a^4}{b^4}$

$2b^4 = a^4$

This implies that a^4 is a multiple of 2, which means that a is a multiple of $2 \Rightarrow a = 2k$

$2b^4 = (2k)^4$

$2b^4 = 16k^4$

$b^4 = 8k^4 = 2(4k^4)$

This implies that b^4 is also a multiple of 2. And in turn, that b is a multiple of 2.

We now have that both a and b are multiples of 2. This contradicts the assumption that a and b have no common factors. Our assumption must be false and the original statement is proven true.

(f) Assume that $\sqrt[4]{3}$ is rational.

Then $\sqrt[4]{3} = \dfrac{a}{b}$ where a and b have no common factors.

$3 = \dfrac{a^4}{b^4}$

$3b^4 = a^4$

This implies that a^4 is a multiple of 3, which means that a is a multiple of $3 \Rightarrow a = 3k$

$3b^4 = (3k)^4$

$3b^4 = 81k^4$

$b^4 = 27k^4 = 3(9k^4)$

This implies that b^4 is also a multiple of 3. And in turn, that b is a multiple of 3.

We now have that both a and b are multiples of 3. This contradicts the assumption that a and b have no common factors. Our assumption must be false and the original statement is proven true.

(g) Assume that $\sqrt[4]{5}$ is rational.

Then $\sqrt[4]{5} = \frac{a}{b}$ where a and b have no common factors.

$5 = \frac{a^4}{b^4}$

$5b^4 = a^4$

This implies that a^4 is a multiple of 5, which means that a is a multiple of $5 \Rightarrow a = 5k$

$5b^4 = (5k)^4$

$5b^4 = 625k^4$

$b^4 = 125k^4 = 5(25k^4)$

This implies that b^4 is also a multiple of 5. And in turn, that b is a multiple of 5.

We now have that both a and b are multiples of 5. This contradicts the assumption that a and b have no common factors. Our assumption must be false and the original statement is proven true.

(h) Assume that $\sqrt[4]{6}$ is rational.

Then $\sqrt[4]{6} = \frac{a}{b}$ where a and b have no common factors.

$6 = \frac{a^4}{b^4}$

$6b^4 = a^4$

This implies that a^4 is a multiple of 6, which means that a is a multiple of $6 \Rightarrow a = 6k$

$6b^4 = (6k)^4$

$6b^4 = 1296k^4$

$b^4 = 216k^4 = 6(36k^4)$

This implies that b^4 is also a multiple of 6. And in turn, that b is a multiple of 6.

We now have that both a and b are multiples of 6. This contradicts the assumption that a and b have no common factors. Our assumption must be false and the original statement is proven true.

3.

(a) Assume that $5\sqrt{3}$ is rational.

Then $5\sqrt{3} = \frac{a}{b}$ where a and b have no common factors and $a, b \in \mathbb{N}$.

$\sqrt{3} = \frac{a}{5b}$

This is a contradiction since we know that $\sqrt{3}$ is irrational,

but as $a, b \in \mathbb{N}$, then $\frac{a}{5b}$ is rational.

Our assumption must be false and the original statement is proven true.

(b) Assume that $2\sqrt{7}$ is rational.

Then $2\sqrt{7} = \frac{a}{b}$ where a and b have no common factors and $a, b \in \mathbb{N}$.

$\sqrt{7} = \frac{a}{2b}$

This is a contradiction since we know that $\sqrt{7}$ is irrational, but as $a, b \in \mathbb{N}$, then $\frac{a}{2b}$ is rational.

Our assumption must be false and the original statement is proven true.

(c) Assume that $2\sqrt{11}$ is rational.

Then $2\sqrt{11} = \frac{a}{b}$ where a and b have no common factors and $a, b \in \mathbb{N}$.

$\sqrt{11} = \frac{a}{2b}$

This is a contradiction since we know that $\sqrt{11}$ is irrational, but as $a, b \in \mathbb{N}$, then $\frac{a}{2b}$ is rational.

Our assumption must be false and the original statement is proven true.

(d) Assume that $4\sqrt{2}$ is rational.

Then $4\sqrt{2} = \frac{a}{b}$ where a and b have no common factors and $a, b \in \mathbb{N}$.

$\sqrt{2} = \frac{a}{4b}$

This is a contradiction since we know that $\sqrt{2}$ is irrational, but as $a, b \in \mathbb{N}$, then $\frac{a}{4b}$ is rational.

Our assumption must be false and the original statement is proven true.

(e) Assume that $4 + \sqrt{3}$ is rational.

Then $4 + \sqrt{3} = \frac{a}{b}$ where a and b have no common factors and $a, b \in \mathbb{N}$.

$\sqrt{3} = \frac{a}{b} - 4b = \frac{a - 4b}{b}$

This is a contradiction since we know that $\sqrt{3}$ is irrational, but as $a - 4b, b \in \mathbb{Z}$, then $\frac{a - 4b}{b}$ is rational.

Our assumption must be false and the original statement is proven true.

(f) Assume that $1 + \sqrt{5}$ is rational.

Then $1 + \sqrt{5} = \frac{a}{b}$ where a and b have no common factors and $a, b \in \mathbb{N}$.

$\sqrt{5} = \frac{a - b}{b}$

This is a contradiction since we know that $\sqrt{5}$ is irrational, but as $a - b, b \in \mathbb{Z}$, then $\frac{a - b}{b}$ is rational.

Our assumption must be false and the original statement is proven true.

(g) Assume that $8 + 2\sqrt{3}$ is rational.

Then $8 + 2\sqrt{5} = \dfrac{a}{b}$ where a and b have no common factors and $a, b \in \mathbb{N}$.

$\sqrt{5} = \dfrac{a - 8b}{2b}$

This is a contradiction since we know that $\sqrt{5}$ is irrational, but as $a - 8b, 2b \in \mathbb{Z}$, then $\dfrac{a - 8b}{2b}$ is rational.

Our assumption must be false and the original statement is proven true.

(h) Assume that $9 + 3\sqrt{11}$ is rational.

Then $9 + 3\sqrt{11} = \dfrac{a}{b}$ where a and b have no common factors and $a, b \in \mathbb{N}$.

$\sqrt{11} = \dfrac{a - 9b}{3b}$

This is a contradiction since we know that $\sqrt{11}$ is irrational, but as $a - 9b, 3b \in \mathbb{Z}$, then $\dfrac{a - 9b}{3b}$ is rational.

Our assumption must be false and the original statement is proven true.

4.

(a) Assume $\log_2 3$ is rational.

Then $\log_2 3 = \dfrac{a}{b}$ where a and b have no common factors and $a, b \in \mathbb{N}$.

$2^{\frac{a}{b}} = 3$

$2^a = 3^b$

This is a contradiction since 2 and 3 are prime numbers, so there is no integer power of 3 equal to an integer power of 2.

Our assumption must be false and the original statement is proven true.

(b) Assume $\log_3 10$ is rational.

Then $\log_3 10 = \dfrac{a}{b}$ where a and b have no common factors and $a, b \in \mathbb{N}$.

$3^{\frac{a}{b}} = 10$

$3^a = 10^b$

$3^a = (2 \times 5)^b$

$3^a = 2^b \times 5^b$

This is a contradiction since 3^a is odd, but $2^b \times 5^b$ is even.

Our assumption must be false and the original statement is proven true.

(c) Assume $\log_4 7$ is rational.

Then $\log_4 7 = \dfrac{a}{b}$ where a and b have no common factors and $a, b \in \mathbb{N}$.

$4^{\frac{a}{b}} = 7$

$4^a = 7^b$

This is a contradiction since 4^a is even, but 7^b is odd.

Our assumption must be false and the original statement is proven true.

(d) Assume $\log_5 6$ is rational.

Then $\log_5 6 = \dfrac{a}{b}$ where a and b have no common factors and $a, b \in \mathbb{N}$.

$5^{\frac{a}{b}} = 6$

$5^a = 6^b$

$5^a = (2 \times 3)^b$

$5^a = 2^b \times 3^b$

This is a contradiction since 5^a is odd, but $2^b \times 3^b$ is even.

Our assumption must be false and the original statement is proven true.

(e) Assume $\log_3 8$ is rational.

Then $\log_3 8 = \dfrac{a}{b}$ where a and b have no common factors and $a, b \in \mathbb{N}$.

$3^{\frac{a}{b}} = 8$

$3^a = 8^b$

$3^a = (2^3)^b$

$3^a = 2^{3b}$

This is a contradiction since 3^a is odd, but 2^{3b} is even.

Our assumption must be false and the original statement is proven true.

(f) Assume $\log_2 5$ is rational.

Then $\log_2 5 = \dfrac{a}{b}$ where a and b have no common factors and $a, b \in \mathbb{N}$.

$2^{\frac{a}{b}} = 5$

$2^a = 5^b$

This is a contradiction since 2^a is even, but 5^b is odd.

Our assumption must be false and the original statement is proven true.

(g) Assume $\log_3 11$ is rational.

Then $\log_3 11 = \dfrac{a}{b}$ where a and b have no common factors and $a, b \in \mathbb{N}$.

$3^{\frac{a}{b}} = 11$

$3^a = 11^b$

This is a contradiction since 3 and 11 are prime numbers, so there is no integer power of 11 equal to an integer power of 3.

Our assumption must be false and the original statement is proven true.

(h) Assume $\log_2 13$ is rational.

Then $\log_2 13 = \dfrac{a}{b}$ where a and b have no common factors and $a, b \in \mathbb{N}$.

$2^{\frac{a}{b}} = 13$

$2^a = 13^b$

This is a contradiction since 2^a is even, but 13^b is odd. 2 and 13 are prime numbers, so there is no integer power of 13 equal to an integer power of 2.

Our assumption must be false and the original statement is proven true.

Exercise 26.4A

In this exercise it is assumed that $n, k \in \mathbb{Z}$.

1. Contrapositive: 'If n is odd, then n^2 is odd'

   $n = 2k + 1$

   $n^2 = (2k + 1)^2$

   $= 4k^2 + 4k + 1$

   $= 2(2k^2 + 2k) + 1$ which is odd.

   Contrapositive is true therefore original statement is true.

2. Contrapositive: 'If n is even, then $n^2 + 1$ is odd'

   $n = 2k$

   $n^2 + 1 = (2k)^2 + 1$

   $= 4k^2 + 1$

   $= 2(2k^2) + 1$ which is odd.

   Contrapositive is true therefore original statement is true.

3. Contrapositive: 'If n is even, then $7n + 1$ is odd'

   $n = 2k$

   $7n + 1 = 7(2k) + 1$

   $= 14k + 1$

   $= 2(7k) + 1$ which is odd.

   Contrapositive is true therefore original statement is true.

4. Contrapositive: 'If n is even, then n^2 is even'

   $n = 2k$

   $n^2 = (2k)^2 = 4k^2$

   $= 2(2k^2)$ which is even.

   Contrapositive is true therefore original statement is true.

5. Contrapositive: 'If n is even, then n^3 is even'

   $n = 2k$

   $n^3 = (2k)^3$

   $= 8k^3$

   $= 2(4k^3)$ which is even.

   Contrapositive is true therefore original statement is true.

6. Contrapositive: 'If n is odd, then n^3 is odd'

   $n = 2k + 1$

   $n^3 = (2k + 1)^3 = 8k^3 + 12k^2 + 6k + 1$

   $= 2(4k^3 + 6k^2 + 3k) + 1$ which is odd.

   Contrapositive is true therefore original statement is true.

7. Contrapositive: 'If n is odd, then $3n^2 - 5$ is even'

   $n = 2k + 1$

   $3n^2 - 5 = 3(2k + 1)^2 - 5$

   $= 12k^2 + 12k - 2$

   $= 2(6k^2 + 6k - 1)$ which is even.

   Contrapositive is true therefore original statement is true.

8. Contrapositive: 'If n is even, then $n^2 - 3n - 3$ is odd'

   $n = 2k$

   $n^2 - 3n - 3 = (2k)^2 - 3(2k) - 3$

   $= 4k^2 - 6k - 3$

   $= 2(2k^2 - 3k - 1) - 1$ which is odd.

   Contrapositive is true therefore original statement is true.

9. Contrapositive: 'If only one of n, m is odd, then $n + m$ is odd'

   $m = 2k + 1, n = 2j$

   $m + n = (2k + 1) + 2j$

   $= 2(2kj + j) + 1$ which is odd.

   Contrapositive is true therefore original statement is true.

10. Contrapositive: 'If only one (or none) of m, n is odd, then mn is even'

    Let $m = 2k + 1, m = 2j + 1$

    $mn = (2k + 1)(2j + 1)$

    $= 4kj + 2k + 2j + 1$

    $= 2(2kj + k + j) + 1$ which is odd.

    Contrapositive is true therefore original statement is true.

    Let $m = 2k, n = 2j$

    $mn = (2k)(2j)$

    $= 4kj$

    $= 2(2kj)$ which is even.

    Contrapositive is true therefore original statement is true.

11. Contrapositive: 'If n, m are both odd, then nm is odd'

    Let $m = 2k + 1, m = 2j + 1$

    $mn = (2k + 1)(2j + 1)$

    $= 4kj + 2k + 2j + 1$

    $= 2(2kj + k + j) + 1$ which is odd.

    Contrapositive is true therefore original statement is true.

Exercise 26.4B

In this exercise it is assumed that $a, b \in \mathbb{Z}$.

1.

(a) Contrapositive statement: 'If both $a < 3$ and $b < 3$, then $a + b < 5$'
Since they are integers, then $a \leq 2$ and $b \leq 2$
$\Rightarrow a + b \leq 4$
$\Rightarrow a + b < 5$
The contrapositive is true, which means that the original statement is also proven true.

(b) Contrapositive statement: 'If both $a < 8$ and $b < 8$, then $a + b < 15$'
Since they are integers, then $a \leq 7$ and $b \leq 7$
$\Rightarrow a + b \leq 14$
$\Rightarrow a + b < 15$
The contrapositive is true, which means that the original statement is also proven true.

(c) Contrapositive statement: 'If both $a < 9$ and $b < 9$, then $a + b < 17$'
Since they are integers, then $a \leq 8$ and $b \leq 8$
$\Rightarrow a + b \leq 16$
$\Rightarrow a + b < 17$
The contrapositive is true, which means that the original statement is also proven true.

(d) Contrapositive statement: 'If both $a < 14$ and $b < 14$, then $a + b < 27$'
Since they are integers, then $a \leq 13$ and $b \leq 13$
$\Rightarrow a + b \leq 26$
$\Rightarrow a + b < 27$
The contrapositive is true, which means that the original statement is also proven true.

(e) Contrapositive statement: 'If both $a < 11$ and $b < 11$, then $a + b < 21$'
Since they are integers, then $a \leq 10$ and $b \leq 10$
$\Rightarrow a + b \leq 20$
$\Rightarrow a + b < 21$
The contrapositive is true, which means that the original statement is also proven true.

(f) Contrapositive statement: 'If both $a < 16$ and $b < 16$, then $a + b < 31$'
Since they are integers, then $a \leq 15$ and $b \leq 15$
$\Rightarrow a + b \leq 30$
$\Rightarrow a + b < 31$
The contrapositive is true, which means that the original

statement is also proven true.

(g) Contrapositive statement: 'If both $a < 21$ and $b < 21$, then $a + b < 41$'
Since they are integers, then $a \leq 20$ and $b \leq 20$
$\Rightarrow a + b \leq 40$
$\Rightarrow a + b < 41$
The contrapositive is true, which means that the original statement is also proven true.

(h) Contrapositive statement: 'If both $a < 25$ and $b < 25$, then $a + b < 49$'
Since they are integers, then $a \leq 24$ and $b \leq 24$
$\Rightarrow a + b \leq 48$
$\Rightarrow a + b < 49$
The contrapositive is true, which means that the original statement is also proven true.

(i) Contrapositive statement: 'If both $a < 27$ and $b < 27$, then $a + b < 53$'
Since they are integers, then $a \leq 26$ and $b \leq 26$
$\Rightarrow a + b \leq 52$
$\Rightarrow a + b < 53$
The contrapositive is true, which means that the original statement is also proven true.

2.

(a) Contrapositive statement: 'If $x = 10$, then $x^3 - 7x^2 + x - 7 \neq 0$'
Sub $x = 10$ into the equation
$10^3 - 7(10)^2 + 10 - 7 = 303 \neq 0$
The contrapositive is true, which means that the original statement is also proven true.

(b) Contrapositive statement: 'If $x = 2$, then $x^3 + x^2 - 17x + 5 \neq 0$'
Sub $x = 2$ into the equation
$2^3 + (2)^2 - 17(2) + 5 = -17 \neq 0$
The contrapositive is true, which means that the original statement is also proven true.

(c) Contrapositive statement: 'If $x = 4$, then $x^3 - 5x^2 + 5x - 25 \neq 0$'
Sub $x = 4$ into the equation
$4^3 - 5(4)^2 + 5(4) - 25 = -21 \neq 0$
The contrapositive is true, which means that the original statement is also proven true.

Exercise 26.5A

1. Show true for $n = 1$
$$LHS = \sum_{r=1}^{1} 1(1 + 1) = 2$$

$RHS = \frac{1}{3}(1)(1 + 1)(1 + 2) = 2$

So result is true when $n = 1$

Assume true for $n = k$

$$\sum_{r=1}^{k} r(r + 1) = \frac{1}{3}k(k + 1)(k + 2)$$

Show also true for $n = k + 1$

Aim: $\sum_{r=1}^{k+1} r(r + 1) = \frac{1}{3}(k + 1)((k + 1) + 1)((k + 1) + 2)$

$= \frac{1}{3}(k + 1)(k + 2)(k + 3)$

$$\sum_{r=1}^{k+1} r(r + 1) = \sum_{r=1}^{k} r(r + 1) + (k + 1)((k + 1) + 1)$$

$= \frac{1}{3}k(k + 1)(k + 2) + (k + 1)(k + 2)$

$= \frac{1}{3}(k + 2)(k + 2)[k + 3]$

$= \frac{1}{3}(k + 1)(k + 2)(k + 3)$ as required.

If the statement is true for $n = k$, then it is true for $n = k + 1$. Since it is true for $n = 1$, by induction it is true for all positive integers n.

2. Show true for $n = 1$

$LHS = \sum_{r=1}^{1} 1(1 + 3) = 4$

$RHS = \frac{1}{3}(1)(1 + 1)(1 + 5) = 4$

So result is true when $n = 1$

Assume true for $n = k$

$$\sum_{r=1}^{k} r(r + 3) = \frac{1}{3}k(k + 1)(k + 5)$$

Show true for $n = k + 1$

Aim: $\sum_{r=1}^{k+1} r(r + 3) = \frac{1}{3}(k + 1)((k + 1) + 1)((k + 1) + 5)$

$= \frac{1}{3}(k + 1)(k + 2)(k + 6)$

$$\sum_{r=1}^{k+1} r(r + 3) = \sum_{r=1}^{k} r(r + 3) + (k + 1)((k + 1) + 3)$$

$= \frac{1}{3}k(k + 1)(k + 5) + (k + 1)(k + 4)$

$= \frac{1}{3}(k + 1)[k(k + 5) + 3(k + 4)]$

$= \frac{1}{3}(k + 1)[k^2 + 8k + 12]$

$= \frac{1}{3}(k + 1)(k + 2)(k + 6)$ as required.

If the statement is true for $n = k$, then it is true for $n = k + 1$. Since it is true for $n = 1$, by induction it is true for all positive integers n.

3. Show true for $n = 1$

$LHS = \sum_{r=1}^{1} 1(2(1) - 1) = 1$

$RHS = \frac{1}{6}(1)(1 + 1)(4(1) - 1) = 1$

So result is true when $n = 1$

Assume true for $n = k$

$$\sum_{r=1}^{k} r(2r - 1) = \frac{1}{6}k(k + 1)(4k - 1)$$

Show true for $n = k + 1$

Aim: $\sum_{r=1}^{k+1} r(2r - 1) = \frac{1}{6}(k + 1)((k + 1) + 1)(4(k + 1) - 1) = \frac{1}{6}(k + 1)(k + 2)(4k + 3)$

$$\sum_{r=1}^{k+1} r(2r - 1) = \sum_{r=1}^{k} r(2r - 1) + (k + 1)(2(k + 1) - 1)$$

$= \frac{1}{6}k(k + 1)(4k - 1) + (k + 1)(2k + 1)$

$= \frac{1}{6}(k + 1)[k(4k - 1) + 6(2k + 1)]$

$= \frac{1}{6}(k + 1)[4k^2 + 11k + 6]$

$= \frac{1}{6}(k + 1)(k + 2)(4k + 3)$ as required.

If the statement is true for $n = k$, then it is true for $n = k + 1$. Since it is true for $n = 1$, by induction it is true for all positive integers n.

4. Show true for $n = 1$

$LHS = \sum_{r=1}^{1} 1(3(1) - 1) = 2$

$RHS = (1)^2(1 + 1) = 2$

So result is true when $n = 1$

Assume true for $n = k$

$$\sum_{r=1}^{k} r(3r - 1) = k^2(k + 1)$$

Show true for $n = k + 1$

Aim: $\sum_{r=1}^{k+1} r(3r - 1) = (k + 1)^2((k + 1) + 1) = (k + 1)^2(k + 2)$

$$\sum_{r=1}^{k+1} r(3r - 1) = \sum_{r=1}^{k} r(3r - 1) + (k + 1)(3(k + 1) - 1)$$

$= k^2(k + 1) + (k + 1)(3k + 2)$

$= (k + 1)[k^2 + (3k + 2)]$

$= (k + 1)[(k + 1)(k + 2)]$

$= (k + 1)^2(k + 2)$ as required.

If the statement is true for $n = k$, then it is true for $n = k + 1$. Since it is true for $n = 1$, by induction it is true for all positive integers n.

5. Show true for $n = 1$

$LHS = \sum_{r=1}^{1} 1(1 + 1)(1 + 2) = 6$

$RHS = \frac{1}{4}(1)(1 + 1)(1 + 2)(1 + 3) = 6$

So result is true when $n = 1$

Assume true for $n = k$

$$\sum_{r=1}^{k} r(r + 1)(r + 2) = \frac{1}{4}k(k + 1)(k + 2)(k + 3)$$

Show true for $n = k + 1$

Aim: $\sum_{r=1}^{k+1} r(r + 1)(r + 2)$

$= \frac{1}{4}(k + 1)((k + 1) + 1)((k + 1) + 2)((k + 1) + 3)$

$= \frac{1}{4}(k + 1)(k + 2)(k + 3)(k + 4)$

$\sum_{r=1}^{k+1} r(r + 1)(r + 2)$

$= \sum_{r=1}^{k} r(r + 1)(r + 2) + (k + 1)[(k + 1) + 1(k + 1) + 2]$

$= \frac{1}{4}k(k + 1)(k + 2)(k + 3) + (k + 1)(k + 2)(k + 3)$

$= \frac{1}{4}(k + 2)(k + 2)(k + 3)[k + 4]$

$= \frac{1}{4}(k + 1)(k + 2)(k + 3)(k + 4)$ as required.

If the statement is true for $n = k$, then it is true for $n = k + 1$. Since it is true for $n = 1$, by induction it is true for all positive integers n.

6. Show true for $n = 1$

$LHS = \sum_{r=1}^{1} 1(1 + 1)(1 + 5) = 12$

$RHS = \frac{1}{4}(1)(1 + 1)(1 + 2)(1 + 7) = 12$

So result is true when $n = 1$

Assume true for $n = k$

$\sum_{r=1}^{k} r(r + 1)(r + 5) = \frac{1}{4}k(k + 1)(k + 2)(k + 7)$

Show true for $n = k + 1$

Aim: $\sum_{r=1}^{k+1} r(r + 1)(r + 5) = \frac{1}{4}(k + 1)((k + 1) + 1)((k + 1) + 2)$

$((k + 1) + 7) = \frac{1}{4}(k + 1)(k + 2)(k + 3)(k + 8)$

$\sum_{r=1}^{k+1} r(r + 1)(r + 5)$

$= \sum_{r=1}^{k} r(r + 1)(r + 5) + (k + 1)((k + 1) + 1)((k + 1) + 5)$

$= \frac{1}{4}k(k + 1)(k + 2)(k + 7) + (k + 1)(k + 2)(k + 6)$

$= \frac{1}{4}(k + 1)(k + 2)[k(k + 7) + 4(k + 6)]$

$= \frac{1}{4}(k + 1)(k + 2)[k^2 + 11k + 24]$

$= \frac{1}{4}(k + 1)(k + 2)(k + 3)(k + 8)$ as required.

If the statement is true for $n = k$, then it is true for $n = k + 1$. Since it is true for $n = 1$, by induction it is true for all positive integers n.

7. Show true for $n = 1$

$LHS = \sum_{r=1}^{1}(1) = 1$ $RHS = \frac{1}{2}(1)(1 + 1) = 1$

So result is true when $n = 1$

Assume true for $n = k$

$\sum_{r=1}^{k} r = \frac{1}{2}k(k + 1)$

Show true for $n = k + 1$

Aim: $\sum_{r=1}^{k+1} r = \frac{1}{2}(k + 1)((k + 1) + 1) = \frac{1}{2}(k + 1)(k + 2)$

$\sum_{r=1}^{k+1}(r) = \sum_{r=1}^{k}(r) + (k + 1)$

$= \frac{1}{2}k(k + 1) + (k + 1)$

$= \frac{1}{2}(k + 1)[k + 2]$

$= \frac{1}{2}(k + 1)(k + 2)$ as required.

If the statement is true for $n = k$, then it is true for $n = k + 1$. Since it is true for $n = 1$, by induction it is true for all positive integers n.

8. Show true for $n = 1$

$LHS = \sum_{r=1}^{1} 1^2 = 1$ $RHS = \frac{1}{6}(1)(1 + 1)(2(1) + 1) = 1$

So result is true when $n = 1$

Assume true for $n = k$

$\sum_{r=1}^{k} r^2 = \frac{1}{6}k(k + 1)(2k + 1)$

Show true for $n = k + 1$

Aim: $\sum_{r=1}^{k+1} r^2 = \frac{1}{6}(k + 1)((k + 1) + 1)(2(k + 1) + 1)$

$= \frac{1}{6}(k + 1)(k + 2)(2k + 3)$

$\sum_{r=1}^{k+1} r^2 = \sum_{r=1}^{k} r^2 + (k + 1)^2$

$= \frac{1}{6}k(k + 1)(2k + 1) + (k + 1)^2$

$= \frac{1}{6}(k + 1)[k(2k + 1) + 6(k + 1)]$

$= \frac{1}{6}(k + 1)[2k^2 + 7k + 6]$

$= \frac{1}{6}(k + 1)(k + 2)(2k + 3)$ as required.

If the statement is true for $n = k$, then it is true for $n = k + 1$. Since it is true for $n = 1$, by induction it is true for all positive integers n.

9. Show true for $n = 1$

$LHS = \sum_{r=1}^{1} 1^3 = 1$ $RHS = \frac{1}{4}(1)^2(1 + 1)^2 = 1$

So result is true when $n = 1$

Assume true for $n = k$

$\sum_{r=1}^{k} r^3 = \frac{1}{4}k^2(k + 1)^2$

Show true for $n = k + 1$

Aim: $\sum_{r=1}^{k+1} r^3 = \frac{1}{4}(k + 1)^2((k + 1) + 1)^2 = \frac{1}{4}(k + 1)^2(k + 2)^2$

$\sum_{r=1}^{k+1} r^3 = \sum_{r=1}^{k} r^3 + (k + 1)^3$

$= \frac{1}{4}k^2(k + 1)^2 + (k + 1)^3$

$= \frac{1}{4}(k + 1)^2[k^2 + 4(k + 1)]$

$= \frac{1}{4}(k + 1)^2[(k + 2)(k + 2)]$

$= \frac{1}{4}(k + 1)^2(k + 2)^2$ as required.

If the statement is true for $n = k$, then it is true for $n = k + 1$. Since it is true for $n = 1$, by induction it is true for all positive integers n.

10. Show true for $n = 1$

$LHS = \sum_{r=1}^{1} 1^2(1 - 1) = 0$

$RHS = \frac{1}{12}(1)(1^2 - 1)(3(1) - 1) = 0$

So result is true when $n = 1$

Assume true for $n = k$

$\sum_{r=1}^{k} r^2(r - 1) = \frac{1}{12}k(k^2 - 1)(3k + 2)$

Show true for $n = k + 1$

Aim: $\sum_{r=1}^{k+1} r^2(r - 1) = \frac{1}{12}(k + 1)((k + 1)^2 - 1)(3(k + 1) + 2)$

$= \frac{1}{12}k(k + 1)(k + 2)(3k + 5)$

$\sum_{r=1}^{k+1} r^2(r - 1) = \sum_{r=1}^{k} r^2(r - 1) + (k + 1)^2((k + 1) - 1)$

$= \frac{1}{12}k(k - 1)(k + 1)(3k + 2) + (k + 1)^2(k)$

$= \frac{1}{12}k(k + 1)[(k - 1)(3k + 2) + 12(k + 1)]$

$= \frac{1}{12}k(k + 1)[3k^2 + 11k + 10]$

$= \frac{1}{12}k(k + 1)(k + 2)(3k + 5)$ as required.

If the statement is true for $n = k$, then it is true for $n = k + 1$. Since it is true for $n = 1$, by induction it is true for all positive integers n.

11. Show true for $n = 1$

$LHS = \sum_{r=1}^{1} 3(1^2 - 1) = 0$

$RHS = (1 - 1)(1)(1 + 1) = 0$

So result is true when $n = 1$

Assume true for $n = k$

$\sum_{r=1}^{k} 3(r^2 - r) = (k - 1)k(k + 1)$

Show true for $n = k + 1$

Aim: $\sum_{r=1}^{k+1} 3(r^2 - r) = ((k + 1) - 1)(k + 1)((k + 1) + 1)$

$= (k)(k + 1)(k + 2)$

$\sum_{r=1}^{k+1} 3(r^2 - r) = \sum_{r=1}^{k} 3(r^2 - r) + 3[(k + 1)^2 - (k + 1)]$

$= (k - 1)k(k + 1) + 3[(k + 1)^2 - (k + 1)]$

$= (k - 1)k(k + 1) + 3[(k - 1)^2 - (k + 1)]$

$= (k - 1)k(k + 1) + 3(k^2 + k)$

$= (k - 1)k(k + 1) + 3k(k + 1)$

$= k(k + 1)[(k - 1) + 3]$

$= k(k + 1)(k + 2)$ as required.

If the statement is true for $n = k$, then it is true for $n = k + 1$. Since it is true for $n = 1$, by induction it is true for all positive integers n.

12. Show true for $n = 1$

$LHS = \sum_{r=1}^{1} 1(2^{1-1}) = 1$

$RHS = 1 + 2^1(1 - 1) = 1$

So result is true when $n = 1$

Assume true for $n = k$

$\sum_{r=1}^{k} r(2^{r-1}) = 1 + 2^k(k - 1)$

Show true for $n = k + 1$

Aim: $\sum_{r=1}^{k+1} r(2^{r-1}) = 1 + 2^{k+1}((k + 1) - 1) = 1 + 2^{k+1}(k)$

$\sum_{r=1}^{k+1} r(2^{r-1}) = \sum_{r=1}^{k} r(2^{r-1}) + (k + 1)(2^{(k+1)-1})$

$= 1 + 2^k(k - 1) + (k + 1)(2^k)$

$= 1 + 2^k[(k - 1) + (k + 1)]$

$= 1 + 2^k(2k)$

$= 1 + k(2^k \times 2)$

$= 1 + (k)2^{k+1}$ as required.

If the statement is true for $n = k$, then it is true for $n = k + 1$. Since it is true for $n = 1$, by induction it is true for all positive integers n.

Exercise 26.5B

1. Show true for $n = 1$

$LHS = \sum_{r=1}^{1} \frac{1}{1(1 + 1)} = \frac{1}{2}$

$RHS = \frac{1}{1 + 1} = \frac{1}{2}$

So result is true when $n = 1$

Assume true for $n = k$

$\sum_{r=1}^{k} \frac{1}{r(r + 1)} = \frac{k}{k + 1}$

Show true for $n = k + 1$

Aim: $\sum_{r=1}^{k+1} \frac{1}{r(r+1)} = \frac{k+1}{(k+1)+1} = \frac{k+1}{k+2}$

$\sum_{r=1}^{k+1} \frac{1}{r(r+1)} = \sum_{r=1}^{k} \frac{1}{r(r+1)} + \frac{1}{(k+1)(k+1)+1}$

$= \frac{k}{k+1} + \frac{1}{(k+1)(k+2)}$

$= \frac{1}{k+1}\left[k + \frac{1}{k+2}\right]$

$= \frac{1}{k+1}\left[\frac{k(k+2)}{k+2} + \frac{1}{k+2}\right]$

$= \frac{1}{k+1}\left[\frac{k^2 + 2k + 1}{k+2}\right]$

$= \frac{1}{k+1}\left[\frac{(k+1)^2}{k+2}\right]$

$= \frac{k+1}{k+2}$ as required.

If the statement is true for $n = k$, then it is true for $n = k + 1$. Since it is true for $n = 1$, by induction it is true for all positive integers n.

2. Show true for $n = 1$

$LHS = \sum_{r=1}^{1} \frac{1}{1(1+1)(1+2)} = \frac{1}{6}$

$RHS = \frac{1^2 + 3(1)}{4(1+1)(1+2)} = \frac{1}{6}$

So result is true when $n = 1$

Assume true for $n = k$

$\sum_{r=1}^{k} \frac{1}{r(r+1)(r+2)} = \frac{k^2 + 3k}{4(k+1)(k+2)}$

Show true for $n = k + 1$

Aim: $\sum_{r=1}^{k+1} \frac{1}{r(r+1)(r+2)}$

$= \frac{(k+1)^2 + 3(k+1)}{4((k+1)+1)((k+1)+2)} = \frac{k^2 + 5k + 4}{4(k+2)(k+3)}$

$\sum_{r=1}^{k+1} \frac{1}{r(r+1)(r+2)} = \sum_{r=1}^{k} \frac{1}{r(r+1)(r+2)}$

$+ \frac{1}{[(k+1)(k+1)+1][(k+1)+2]}$

$= \frac{k^2 + 3k}{4(k+1)(k+2)} + \frac{1}{(k+1)(k+2)(k+3)}$

$= \frac{1}{4(k+1)(k+2)}\left[(k^2 + 3k) + \frac{4}{k+3}\right]$

$= \frac{1}{4(k+1)(k+2)}\left[\frac{(k^2 + 3k)(k+3)}{k+3} + \frac{4}{k+3}\right]$

$= \frac{1}{4(k+1)(k+2)}\left[\frac{k^3 + 6k^2 + 9k + 4}{k+3}\right]$

$= \frac{1}{4(k+1)(k+2)}\left[\frac{(k+1)(k^2 + 5k + 4)}{k+3}\right]$

$= \frac{k^2 + 5k + 4}{4(k+2)(k+3)}$ as required.

If the statement is true for $n = k$, then it is true for

$n = k + 1$. Since it is true for $n = 1$, by induction it is true for all positive integers n.

3. Show true for $n = 1$

$LHS = \sum_{r=1}^{1} \frac{1}{(1+1)!} = \frac{1}{2}$

$RHS = 1 - \frac{1}{(1+1)!} = \frac{1}{2}$

So result is true when $n = 1$

Assume true for $n = k$

$\sum_{r=1}^{k} \frac{1}{r(r+1)!} = 1 - \frac{1}{(k+1)!}$

Show true for $n = k + 1$

Aim: $\sum_{r=1}^{k+1} \frac{r}{(r+1)!} = 1 - \frac{1}{((k+1)+1)!} = 1 - \frac{1}{(k+2)!}$

$\sum_{r=1}^{k+1} \frac{r}{(r+1)!} = \sum_{r=1}^{k} \frac{r}{(r+1)!} + \frac{k+1}{((k+1)+1)!}$

$= 1 - \frac{1}{(k+1)!} + \frac{k+1}{(k+2)!}$

$= 1 - \frac{k+2}{(k+2)(k+1)!} + \frac{(k+1)(k+1)}{(k+1)(k+2)}$

$= 1 - \left[\frac{k^2 + 2k + 1 - (k+2)}{(k+1)(k+2)}\right]$

$= 1 - \frac{1}{(k+2)!}$

$=$ as required.

If the statement is true for $n = k$, then it is true for $n = k + 1$. Since it is true for $n = 1$, by induction it is true for all positive integers n.

4. Show true for $n = 1$

$LHS = \sum_{r=1}^{1} \frac{1-1}{1!} = 0 \quad RHS = \frac{1! - 1}{1!} = 0$

So result is true when $n = 1$

Assume true for $n = k$

$\sum_{r=1}^{k} \frac{r-1}{r!} = \frac{k! - 1}{k!}$

Show true for $n = k + 1$

Aim: $\sum_{r=1}^{k+1} \frac{r-1}{r!} = \frac{(k+1)! - 1}{(k+1)!}$

$\sum_{r=1}^{k+1} \frac{r-1}{r!} = \sum_{r=1}^{k} \frac{r-1}{r!} + \frac{(k+1)! - 1}{(k+1)!}$

$= \frac{k! - 1}{k!} + \frac{k}{(k+1)!}$

$= \frac{(k! - 1)(k+1)}{(k+1)!} + \frac{k}{(k+1)!}$

$= \frac{k!(k+1) - (k+1) + k}{(k+1)!}$

$= \frac{(k+1)! - 1}{(k+1)!}$ as required.

If the statement is true for $n = k$, then it is true for $n = k + 1$. Since it is true for $n = 1$, by induction it is true for all positive integers n.

5. Show true for $n = 1$

$LHS = \sum_{r=1}^{1} 1! \times 1 = 1 \qquad RHS = (1 + 1)! - 1 = 1$

So result is true when $n = 1$

Assume true for $n = k$

$\sum_{r=1}^{k} r!r = (k + 1)! - 1$

Show true for $n = k + 1$

Aim: $\sum_{r=1}^{k+1} r!r = ((k + 1) + 1)! - 1) = (k + 2)! - 1$

$\sum_{r=1}^{k+1} r!r = \sum_{r=1}^{k} r!r + (k + 1)!(k + 1)$

$= (k + 1)! - 1 + (k + 1)!(k + 1)$

$= (k + 1)![1 + (k + 1)] - 1$

$= (k + 1)!(k + 2) - 1$

$= (k + 2)! - 1$ as required.

If the statement is true for $n = k$, then it is true for $n = k + 1$. Since it is true for $n = 1$, by induction it is true for all positive integers n.

6. Show true for $n = 1$

$LHS = \sum_{r=1}^{1} (1^2 + 1)(1)! = 2 \qquad RHS = 1(1 + 1)! = 2$

So result is true when $n = 1$

Assume true for $n = k$

$\sum_{r=1}^{k} (r^2 + 1)r! = k(k + 1)!$

Show true for $n = k + 1$

Aim: $\sum_{r=1}^{k+1} (r^2 + 1)r! = (k + 1)((k + 1) + 1)! = (k + 1)(k + 2)!$

$\sum_{r=1}^{k+1} (r^2 + 1)r! = \sum_{r=1}^{k} (r^2 + 1)r! + ((k + 1)^2 + 1)(k + 1)!$

$= k(k + 1)! + ((k + 1)^2 + 1)(k + 1)!$

$= (k + 1)![k + ((k + 1)^2 + 1)]$

$= (k + 1)![k^2 + 3k + 2]$

$= (k + 1)!(k + 2)(k + 1)$

$= (k + 1)(k + 2)!$ as required.

If the statement is true for $n = k$, then it is true for $n = k + 1$. Since it is true for $n = 1$, by induction it is true for all positive integers n.

7. Show true for $n = 1$

$LHS = \sum_{r=1}^{1} \frac{1}{3^1} = \frac{1}{3}$

$RHS = \frac{1}{2} - \frac{1}{2 \times 3^1} = \frac{1}{2} - \frac{1}{6} = \frac{1}{3}$

So result is true when $n = 1$

Assume true for $n = k$

$\sum_{r=1}^{k} \frac{1}{3^r} = \frac{1}{2} - \frac{1}{2 \times 3^k}$

Show true for $n = k + 1$

Aim: $\sum_{r=1}^{k+1} \frac{1}{3^r} = \frac{1}{2} - \frac{1}{2 \times 3^{k+1}}$

$\sum_{r=1}^{k+1} \frac{1}{3^r} = \sum_{r=1}^{k} \frac{1}{3^r} + \frac{1}{3^{k+1}}$

$= \frac{1}{2} - \frac{1}{2 \times 3^k} + \frac{1}{3^{k+1}}$

$= \frac{1}{2} - \frac{3}{3 \times 2 \times 3^k} + \frac{2}{2 \times 3^{k+1}}$

$= \frac{1}{2} - \left[\frac{3 - 2}{2 \times 3^{k+1}} \right]$

$= \frac{1}{2} - \frac{1}{2 \times 3^{k+1}}$ as required.

If the statement is true for $n = k$, then it is true for $n = k + 1$. Since it is true for $n = 1$, by induction it is true for all positive integers n.

8. Show true for $n = 1$

$LHS = \sum_{r=1}^{1} \frac{1}{2^1} = \frac{1}{2} \qquad RHS = 2 - \frac{1 + 2}{2^1} = \frac{1}{2}$

So result is true when $n = 1$

Assume true for $n = k$

$\sum_{r=1}^{k} \frac{r}{2^r} = 2 - \frac{k + 2}{2^k}$

Show true for $n = k + 1$

Aim: $\sum_{r=1}^{k+1} \frac{r}{2^r} = 2 - \frac{(k + 1) + 2}{2^{k+1}} = 2 - \frac{k + 3}{2^{k+1}}$

$\sum_{r=1}^{k+1} \frac{r}{2^r} = \sum_{r=1}^{k} \frac{r}{2^r} + \frac{k + 1}{2^{k+1}}$

$= 2 - \frac{k + 2}{2^k} + \frac{k + 1}{2^{k+1}}$

$= 2 - \frac{2(k + 2)}{2 \times 2^k} + \frac{k + 1}{2^{k+1}}$

$= 2 - \frac{2(k + 2)}{2^{k+1}} + \frac{k + 1}{2^{k+1}}$

$= 2 - \left[\frac{2k + 4 - (k + 1)}{2^{k+1}} \right]$

$= 2 - \frac{k + 3}{2^{k+1}}$ as required.

If the statement is true for $n = k$, then it is true for $n = k + 1$. Since it is true for $n = 1$, by induction it is true for all positive integers n.

9. Show true for $n = 1$

$LHS = \displaystyle\sum_{r=1}^{1} \frac{1}{4(1)^2 - 1} = \frac{1}{3}$ $RHS = \dfrac{1}{2(1) + 1} = \dfrac{1}{3}$

So result is true when $n = 1$

Assume true for $n = k$

$\displaystyle\sum_{r=1}^{k} \frac{1}{4r^2 - 1} = \frac{k}{2k + 1}$

Show true for $n = k + 1$

Aim: $\displaystyle\sum_{r=1}^{k+1} \frac{1}{4r^2 - 1} = \frac{k + 1}{2(k + 1) + 1} = \frac{k + 1}{2k + 3}$

$\displaystyle\sum_{r=1}^{k+1} \frac{1}{4r^2 - 1} = \sum_{r=1}^{k} \frac{1}{4r^2 - 1} + \frac{1}{4(k + 1)^2 - 1}$

$= \dfrac{k}{2k + 1} + \dfrac{1}{(2k + 1)(2k + 3)}$

$= \dfrac{k(2k + 3)}{(2k + 1)(2k + 3)} + \dfrac{1}{(2k + 1)(2k + 3)}$

$= \dfrac{(2k + 1)(k + 3)}{(2k + 1)(2k + 3)}$

$= \dfrac{k + 1}{2k + 3}$ as required.

If the statement is true for $n = k$, then it is true for $n = k + 1$. Since it is true for $n = 1$, by induction it is true for all positive integers n.

10. Show true for $n = 1$

$LHS = \displaystyle\sum_{r=1}^{1} \frac{2(1)^2 - 1}{1^2(1 + 1)^2} = \frac{1}{4}$ $RHS = \dfrac{1^2}{(1 + 1)^2} = \dfrac{1}{4}$

So result is true when $n = 1$

Assume true for $n = k$

$\displaystyle\sum_{r=1}^{k} \frac{2r^2 - 1}{r^2(r + 1)^2} = \frac{k^2}{(k + 1)^2}$

Show true for $n = k + 1$

Aim: $\displaystyle\sum_{r=1}^{k+1} \frac{2r^2 - 1}{r^2(r + 1)^2} = \frac{(k + 1)^2}{((k + 1) + 1)^2} = \frac{(k + 1)^2}{(k + 2)^2}$

$\displaystyle\sum_{r=1}^{k+1} \frac{2r^2 - 1}{r^2(r + 1)^2} = \sum_{r=1}^{k} \frac{2r^2 - 1}{r^2(r + 1)^2} + \frac{2(k + 1)^2 - 1}{(k + 1)^2((k + 1) + 1)^2}$

$= \dfrac{k^2}{(k + 1)^2} + \dfrac{2k^2 + 4k + 1}{(k + 1)^2(k + 2)^2}$

$= \dfrac{k^2(k + 2)^2}{(k + 1)^2(k + 2)^2} + \dfrac{2k^2 + 4k + 1}{(k + 1)^2(k + 2)^2}$

$= \dfrac{k^2(k + 2)^2 + 2k^2 + 4k + 1}{(k + 1)^2(k + 2)^2}$

$= \dfrac{(k + 1)^4}{(k + 1)^2(k + 2)^2}$

$= \dfrac{(k + 1)^2}{(k + 2)^2}$ as required.

If the statement is true for $n = k$, then it is true for $n = k + 1$. Since it is true for $n = 1$, by induction it is true for all positive integers n.

11. Show true for $n = 1$

$LHS = \displaystyle\sum_{r=1}^{1} \frac{1 \times 2^1}{(1 + 2)!} = \frac{1}{2}$ $RHS = 1 - \dfrac{2^{1+1}}{(1 + 2)!} = \dfrac{1}{3}$

So result is true when $n = 1$

Assume true for $n = k$

$\displaystyle\sum_{r=1}^{k} \frac{r \times 2^r}{(r + 2)!} = 1 - \frac{2^{k+1}}{(k + 2)!}$

Show true for $n = k + 1$

Aim: $\displaystyle\sum_{r=1}^{k+1} \frac{r \times 2^r}{(r + 2)!} = 1 - \frac{2^{(k+1)+1}}{((k + 1) + 2)!} = 1 - \frac{2^{k+2}}{(k + 3)!}$

$\displaystyle\sum_{r=1}^{k+1} \frac{r \times 2^r}{(r + 2)!} = \sum_{r=1}^{k} \frac{r \times 2^r}{(r + 2)!} + \frac{(k + 1) \times 2^{(k+1)}}{((k + 1) + 2)!}$

$= 1 - \dfrac{2^{k+1}}{(k + 2)!} + \dfrac{(k + 1) \times 2^{(k+1)}}{(k + 3)!}$

$= 1 - \dfrac{2^{k+1}(k + 3)}{(k + 2)!(k + 3)} + \dfrac{(k + 1) \times 2^{(k+1)}}{(k + 3)!}$

$= 1 - \left[\dfrac{2^{k+1}(k + 3) - (k + 1) \times 2^{(k+1)}}{(k + 2)!(k + 3)}\right]$

$= 1 - \dfrac{2^{k+1}(k + 3) - (k + 1)}{(k + 2)!(k + 3)}$

$= 1 - \dfrac{2^{k+1} \times 2^1}{(k + 2)!(k + 3)}$

$= 1 - \dfrac{2^{k+2}}{(k + 3)!}$ as required.

If the statement is true for $n = k$, then it is true for $n = k + 1$. Since it is true for $n = 1$, by induction it is true for all positive integers n.

12. Show true for $n = 1$

$LHS = \displaystyle\sum_{r=1}^{1} \frac{3(1)^2 + 2}{1(1 + 1)(1 + 2)^2} = \frac{5}{6}$

$RHS = \dfrac{1(2(1) + 3)}{(1 + 1)(1 + 2)} = \dfrac{5}{6}$

So result is true when $n = 1$

Assume true for $n = k$

$\displaystyle\sum_{r=1}^{k} \frac{3r + 2}{r(r + 1)(r + 2)} = \frac{k(2k + 3)}{(k + 1)(k + 2)}$

Show true for $n = k + 1$

Aim: $\displaystyle\sum_{r=1}^{k+1} \frac{3r + 2}{r(r + 1)(r + 2)} = \frac{(k + 1)(2(k + 1) + 3)}{((k + 1) + 1)((k + 1) + 2)}$

$= \dfrac{(k + 1)(2k + 5)}{(k + 2)(k + 3)}$

$\displaystyle\sum_{r=1}^{k+1} \frac{3r + 2}{r(r + 1)(r + 2)} = \sum_{r=1}^{k} \frac{3r + 2}{r(r + 1)(r + 2)}$

$+ \dfrac{(k + 1)((k + 1) + 1)((k + 1) + 2)}{}$

$= \dfrac{k(2k + 3)}{(k + 1)(k + 2)} + \dfrac{3k + 5}{(k + 1)(k + 2)(k + 3)}$

$= \dfrac{k(2k + 3)(k + 3)}{(k + 1)(k + 2)(k + 3)} + \dfrac{3k + 5}{(k + 1)(k + 2)(k + 3)}$

$$= \frac{k(2k + 3)(k + 3) + 3k + 5}{(k + 1)(k + 2)(k + 3)}$$

$$= \frac{(k + 1)(k + 1)(2k + 5)}{(k + 1)(k + 2)(k + 3)}$$

$$= \frac{(k + 1)(2k + 5)}{(k + 2)(k + 3)} \text{ as required.}$$

If the statement is true for $n = k$, then it is true for $n = k + 1$. Since it is true for $n = 1$, by induction it is true for all positive integers n.

Exercise 26.5C

1. Show true for $n = 1$

   $7^1 - 1 = 6 = 6(1)$ which is divisible by 6.

   Hence the statement is true for $n = 1$

   Assume true for $n = k$

   $7^k - 1 = 6m$ where $m \in \mathbb{Z}^+$

   Show true for $n = k + 1$

   Aim: $7^{k+1} - 1 = 6r$ where $r \in \mathbb{Z}^+$

   $7^{k+1} - 1 = 7^k \times 7^1 - 1$

   $7^{k+1} - 1 = 7^1(7^k) - 1$

   $7^{k+1} - 1 = 7(7^k - 1 + 1) - 1$

   $7^{k+1} - 1 = 7(6m + 1) - 1$

   $7^{k+1} - 1 = 7 \times 6m + 7 \times 1 - 1$

   $7^{k+1} - 1 = 6 \times 7m + 6$

   $7^{k+1} - 1 = 6(7m + 1)$ which is divisible by 6

   If the statement is true for $n = k$, then it is true for $n = k + 1$. Since it is true for $n = 1$, by induction it is true for all positive integers n.

2. Show true for $n = 1$

   $4^1 - 1 = 3 = 3(1)$ which is divisible by 3.

   Hence the statement is true for $n = 1$

   Assume true for $n = k$

   $4^k - 1 = 3m$ where $m \in \mathbb{Z}^+$

   Show true for $n = k + 1$

   Aim: $4^{k+1} - 1 = 3r$ where $r \in \mathbb{Z}^+$

   $4^{k+1} - 1 = 4^k \times 4^1 - 1$

   $4^{k+1} - 1 = 4^1(4^k) - 1$

   $4^{k+1} - 1 = 4(4^k - 1 + 1) - 1$

   $4^{k+1} - 1 = 4(3m + 1) - 1$

   $4^{k+1} - 1 = 4 \times 3m + 4 \times 1 - 1$

   $4^{k+1} - 1 = 3 \times 4m + 3$

   $4^{k+1} - 1 = 3(4m + 1)$ which is divisible by 3

   If the statement is true for $n = k$, then it is true for $n = k + 1$. Since it is true for $n = 1$, by induction it is true for all positive integers n.

3. Show true for $n = 1$

   $5^1 + 3 = 8 = 4(2)$ which is divisible by 4.

   Hence the statement is true for $n = 1$

   Assume true for $n = k$

   $5^k + 3 = 4m$ where $m \in \mathbb{Z}^+$

   Show true for $n = k + 1$

   Aim: $5^{k+1} + 3 = 4r$ where $r \in \mathbb{Z}^+$

   $5^{k+1} + 3 = 5^k \times 5^1 + 3$

   $5^{k+1} + 3 = 5^1(5^k) + 3$

   $5^{k+1} + 3 = 5(5^k + 3 - 3) + 3$

   $5^{k+1} + 3 = 5(4m - 3) + 3$

   $5^{k+1} + 3 = 5 \times 4m - 5 \times 3 + 3$

   $5^{k+1} + 3 = 4 \times 5m - 12$

   $5^{k+1} + 3 = 4(5m - 3)$ which is divisible by 4

   If the statement is true for $n = k$, then it is true for $n = k + 1$. Since it is true for $n = 1$, by induction it is true for all positive integers n.

4. Show true for $n = 1$

   $6^1 + 4 = 10 = 5(2)$ which is divisible by 5.

   Hence the statement is true for $n = 1$

   Assume true for $n = k$

   $6^k + 4 = 5m$ where $m \in \mathbb{Z}^+$

   Show true for $n = k + 1$

   Aim: $6^{k+1} + 4 = 5r$ where $r \in \mathbb{Z}^+$

   $6^{k+1} + 4 = 6^k \times 6^1 + 4$

   $6^{k+1} + 4 = 6^1(6^k) + 4$

   $6^{k+1} + 4 = 6(6^k + 4 - 4) + 4$

   $6^{k+1} + 4 = 6(5m - 4) + 4$

   $6^{k+1} + 4 = 6 \times 5m - 6 \times 4 + 4$

   $6^{k+1} + 4 = 5 \times 6m - 20$

   $6^{k+1} + 4 = 5(6m - 4)$ which is divisible by 5

   If the statement is true for $n = k$, then it is true for $n = k + 1$. Since it is true for $n = 1$, by induction it is true for all positive integers n.

5. Show true for $n = 1$

   $5^2 + 7 = 32 = 8(4)$ which is divisible by 8.

   Hence the statement is true for $n = 1$

   Assume true for $n = k$

   $5^{2k} + 7 = 8m$ where $m \in \mathbb{Z}^+$

   Show true for $n = k + 1$

   Aim: $5^{2k+2} + 7 = 8r$ where $r \in \mathbb{Z}^+$

   $5^{2k+2} + 7 = 5^{2k} \times 5^2 + 7$

   $5^{2k+2} + 7 = 5^2(5^{2k}) + 7$

   $5^{2k+2} + 7 = 25(5^{2k} + 7 - 7) + 7$

$5^{2k+2} + 7 = 25(8m - 7) + 7$

$5^{2k+2} + 7 = 25 \times 8m - 25 \times 7 + 7$

$5^{2k+2} + 7 = 8 \times 25m - 168$

$5^{2k+2} + 7 = 8(25m - 21)$ which is divisible by 8

If the statement is true for $n = k$, then it is true for $n = k + 1$. Since it is true for $n = 1$, by induction it is true for all positive integers n.

6. Show true for $n = 1$

$2^2 - 1 = 3 = 3(1)$ which is divisible by 3.

Hence the statement is true for $n = 1$

Assume true for $n = k$

$2^{2k} - 1 = 3m$ where $m \in \mathbb{Z}^+$

Show true for $n = k + 1$

Aim: $2^{2k+2} - 1 = 3r$ where $r \in \mathbb{Z}^+$

$2^{2k+2} - 1 = 2^{2k} \times 2^2 - 1$

$2^{2k+2} - 1 = 2^2(2^{2k}) - 1$

$2^{2k+2} - 1 = 4(2^{2k} - 1 + 1) - 1$

$2^{2k+2} - 1 = 4(3m + 1) - 1$

$2^{2k+2} - 1 = 4 \times 3m + 4 \times 1 - 1$

$2^{2k+2} - 1 = 3 \times 4m + 3$

$2^{2k+2} - 1 = 3(4m + 1)$ which is divisible by 3

If the statement is true for $n = k$, then it is true for $n = k + 1$. Since it is true for $n = 1$, by induction it is true for all positive integers n.

7. Show true for $n = 1$

$4^2 - 1 = 15 = 15(1)$ which is divisible by 15.

Hence the statement is true for $n = 1$

Assume true for $n = k$

$4^{2k} - 1 = 15m$ where $m \in \mathbb{Z}^+$

Show true for $n = k + 1$

Aim: $4^{2k+2} - 1 = 15r$ where $r \in \mathbb{Z}^+$

$4^{2k+2} - 1 = 4^{2k} \times 4^2 - 1$

$4^{2k+2} - 1 = 4^2(4^{2k}) - 1$

$4^{2k+2} - 1 = 16(4^{2k} - 1 + 1) - 1$

$4^{2k+2} - 1 = 16(15m + 1) - 1$

$4^{2k+2} - 1 = 16 \times 15m + 16 \times 1 - 1$

$4^{2k+2} - 1 = 15 \times 16m + 15$

$4^{2k+2} - 1 = 15(16m + 1)$ which is divisible by 15

If the statement is true for $n = k$, then it is true for

$n = k + 1$. Since it is true for $n = 1$, by induction it is true for all positive integers n.

8. Show true for $n = 1$

$3^2 - 1 = 8 = 8(1)$ which is divisible by 8.

Hence the statement is true for $n = 1$

Assume true for $n = k$

$3^{2k} - 1 = 8m$ where $m \in \mathbb{Z}^+$

Show true for $n = k + 1$

Aim: $3^{2k+2} - 1 = 8r$ where $r \in \mathbb{Z}^+$

$3^{2k+2} - 1 = 3^{2k} \times 3^2 - 1$

$3^{2k+2} - 1 = 3^2(3^{2k}) - 1$

$3^{2k+2} - 1 = 9(3^{2k} - 1 + 1) - 1$

$3^{2k+2} - 1 = 9(8m + 1) - 1$

$3^{2k+2} - 1 = 9 \times 8m + 9 \times 1 - 1$

$3^{2k+2} - 1 = 8 \times 9m + 8$

$3^{2k+2} - 1 = 8(9m + 1)$ which is divisible by 8

If the statement is true for $n = k$, then it is true for $n = k + 1$. Since it is true for $n = 1$, by induction it is true for all positive integers n.

9. Show true for $n = 1$

$3^2 + 7 = 16 = 8(2)$ which is divisible by 8.

Hence the statement is true for $n = 1$

Assume true for $n = k$

$3^{2k} + 7 = 8m$ where $m \in \mathbb{Z}^+$

Show true for $n = k + 1$

Aim: $3^{2k+2} + 7 = 8r$ where $r \in \mathbb{Z}^+$

$3^{2k+2} + 7 = 3^{2k} \times 3^2 + 7$

$3^{2k+2} + 7 = 3^2(3^{2k}) + 7$

$3^{2k+2} + 7 = 9(3^{2k} + 7 - 7) + 7$

$3^{2k+2} + 7 = 9(8m - 7) + 7$

$3^{2k+2} + 7 = 9 \times 8m - 9 \times 7 + 7$

$3^{2k+2} + 7 = 8 \times 9m - 56$

$3^{2k+2} + 7 = 8(9m - 21)$ which is divisible by 8

If the statement is true for $n = k$, then it is true for $n = k + 1$. Since it is true for $n = 1$, by induction it is true for all positive integers n.

10. Show true for $n = 1$

$8^1 - 3^1 = 5 = 5(1)$ which is divisible by 5.

Hence the statement is true for $n = 1$

Assume true for $n = k$

$8^k - 3^k = 5m$ where $m \in \mathbb{Z}^+$

Show true for $n = k + 1$

Aim: $8^{k+1} - 3^{k+1} = 5r$ where $r \in \mathbb{Z}^+$

$8^{k+1} - 3^{k+1} = 8^k \times 8^1 - 3^k \times 3^1$

$8^{k+1} - 3^{k+1} = 8^1(8^k) - 3^k \times 3$

$8^{k+1} - 3^{k+1} = 8(8^k - 3^k + 3^k) - 3^k \times 3$

$8^{k+1} - 3^{k+1} = 8(5m + 3^k) - 3^k \times 3$

$8^{k+1} - 3^{k+1} = 8 \times 5m + 8 \times 3^k - 3^k \times 3$

$8^{k+1} - 3^{k+1} = 5 \times 8m + 3^k(8 - 3)$

$8^{k+1} - 3^{k+1} = 5 \times 8m + 3^k(5)$

$8^{k+1} - 3^{k+1} = 5(8m + 3^k)$ which is divisible by 5

If the statement is true for $n = k$, then it is true for $n = k + 1$. Since it is true for $n = 1$, by induction it is true for all positive integers n.

11. Show true for $n = 1$
$9^1 - 2^1 = 7 = 7(1)$ which is divisible by 7.
Hence the statement is true for $n = 1$

Assume true for $n = k$
$9^k - 2^k = 7m$ where $m \in \mathbb{Z}^+$

Show true for $n = k + 1$
Aim: $9^{k+1} - 2^{k+1} = 7r$ where $r \in \mathbb{Z}^+$

$9^{k+1} - 2^{k+1} = 9^k \times 9^1 - 2^k \times 2^1$
$9^{k+1} - 2^{k+1} = 9^1(9^k) - 2^k \times 2$
$9^{k+1} - 2^{k+1} = 9(9^k - 2^k + 2^k) - 2^k \times 2$
$9^{k+1} - 2^{k+1} = 9(7m + 2^k) - 2^k \times 2$
$9^{k+1} - 2^{k+1} = 9 \times 7m + 9 \times 2^k - 2^k \times 2$
$9^{k+1} - 2^{k+1} = 7 \times 9m + 2^k(9 - 2)$
$9^{k+1} - 2^{k+1} = 7 \times 9m + 2^k(7)$
$9^{k+1} - 2^{k+1} = 7(9m + 2^k)$ which is divisible by 7

If the statement is true for $n = k$, then it is true for $n = k + 1$. Since it is true for $n = 1$, by induction it is true for all positive integers n.

12. Show true for $n = 1$
$9^1 - 5^1 = 4 = 4(1)$ which is divisible by 4.
Hence the statement is true for $n = 1$

Assume true for $n = k$
$9^k - 5^k = 4m$ where $m \in \mathbb{Z}^+$

Show true for $n = k + 1$
Aim: $9^{k+1} - 5^{k+1} = 4r$ where $r \in \mathbb{Z}^+$

$9^{k+1} - 5^{k+1} = 9^k \times 9^1 - 5^k \times 5^1$
$9^{k+1} - 5^{k+1} = 9^1(9^k) - 5^k \times 5$
$9^{k+1} - 5^{k+1} = 9(9^k - 5^k + 5^k) - 5^k \times 5$
$9^{k+1} - 5^{k+1} = 9(4m + 5^k) - 5^k \times 5$
$9^{k+1} - 5^{k+1} = 9 \times 4m + 9 \times 5^k - 5^k \times 5$
$9^{k+1} - 5^{k+1} = 4 \times 9m + 5^k(9 - 5)$
$9^{k+1} - 5^{k+1} = 4 \times 9m + 5^k(4)$
$9^{k+1} - 5^{k+1} = 4(9m + 5^k)$ which is divisible by 4

If the statement is true for $n = k$, then it is true for $n = k + 1$. Since it is true for $n = 1$, by induction it is true for all positive integers n.

13. Show true for $n = 1$
$2^1 + 6^1 = 8 = 8(1)$ which is divisible by 8.
Hence the statement is true for $n = 1$

Assume true for $n = k$
$2^k + 6^k = 8m$ where $m \in \mathbb{Z}^+$

Show true for $n = k + 1$
Aim: $2^{k+1} + 6^{k+1} = 8r$ where $r \in \mathbb{Z}^+$

$2^{k+1} + 6^{k+1} = 2^k \times 2^1 + 6^k \times 6^1$
$2^{k+1} + 6^{k+1} = 2^1(2^k) + 6^k \times 6$
$2^{k+1} + 6^{k+1} = 2(2^k + 6^k - 6^k) + 6^k \times 6$
$2^{k+1} + 6^{k+1} = 2(8m + 6^k) + 6^k \times 6$
$2^{k+1} + 6^{k+1} = 2 \times 8m + 2 \times 6^k + 6^k \times 6$
$2^{k+1} + 6^{k+1} = 8 \times 2m + 6^k(2 + 6)$
$2^{k+1} + 6^{k+1} = 8 \times 2m + 6^k(8)$
$2^{k+1} + 6^{k+1} = 8(2m + 6^k)$ which is divisible by 8

If the statement is true for $n = k$, then it is true for $n = k + 1$. Since it is true for $n = 1$, by induction it is true for all positive integers n.

14. Show true for $n = 1$
$8^1 - 2^1 = 6 = 6(1)$ which is divisible by 6.
Hence the statement is true for $n = 1$

Assume true for $n = k$
$8^k - 2^k = 6m$ where $m \in \mathbb{Z}^+$

Show true for $n = k + 1$
Aim: $8^{k+1} - 2^{k+1} = 6r$ where $r \in \mathbb{Z}^+$

$8^{k+1} - 2^{k+1} = 8^k \times 8^1 - 2^k \times 2^1$
$8^{k+1} - 2^{k+1} = 8^1(8^k) - 2^k \times 2$
$8^{k+1} - 2^{k+1} = 8(8^k - 2^k + 2^k) - 2^k \times 2$
$8^{k+1} - 2^{k+1} = 8(6m + 2^k) - 2^k \times 2$
$8^{k+1} - 2^{k+1} = 8 \times 6m + 8 \times 2^k - 2^k \times 2$
$8^{k+1} - 2^{k+1} = 6 \times 8m + 2^k(8 - 2)$
$8^{k+1} - 2^{k+1} = 6 \times 8m + 2^k(6)$
$8^{k+1} - 2^{k+1} = 6(8m + 2^k)$ which is divisible by 5

If the statement is true for $n = k$, then it is true for $n = k + 1$. Since it is true for $n = 1$, by induction it is true for all positive integers n.

15. Show true for $n = 1$
$7^1 + 4^1 + 1 = 12 = 6(2)$ which is divisible by 6.
Hence the statement is true for $n = 1$

Assume true for $n = k$
$7^k + 4^k + 1 = 6m$ where $m \in \mathbb{Z}^+$

Show true for $n = k + 1$
Aim: $7^{k+1} + 4^{k+1} + 1 = 6r$ where $r \in \mathbb{Z}^+$

$7^{k+1} + 4^{k+1} + 1 = 7^k \times 7^1 + 4^k \times 4^1 + 1$
$7^{k+1} + 4^{k+1} + 1 = 7^1(7^k) + 4^k \times 4 + 1$

$7^{k+1} + 4^{k+1} + 1 = 7(7^k + 4^k + 1 - 4^k - 1) + 4^k \times 4 + 1$

$7^{k+1} + 4^{k+1} + 1 = 7(6m - 4^k - 1) + 4^k \times 4 + 1$

$7^{k+1} + 4^{k+1} + 1 = 7 \times 6m - 7 \times 4^k - 7 \times 1 + 4^k \times 4 + 1$

$7^{k+1} + 4^{k+1} + 1 = 6 \times 7m + 4^k(-7 + 4) + (-7 + 1)$

$7^{k+1} + 4^{k+1} + 1 = 6 \times 7m + 4^k(-3) - 6$

$7^{k+1} + 4^{k+1} + 1 = 6 \times 7m + (2^2)^k(-3) - 6$

$7^{k+1} + 4^{k+1} + 1 = 6 \times 7m + 2^{2k}(-3) - 6$

$7^{k+1} + 4^{k+1} + 1 = 6 \times 7m + 2^1 \times 2^{2k-1}(-3) - 6$

$7^{k+1} + 4^{k+1} + 1 = 6 \times 7m + 2^{2k-1}(-6) - 6$

$7^{k+1} + 4^{k+1} + 1 = 6(7m - 2^{2k-1} - 1)$ which is divisible by 6

If the statement is true for $n = k$, then it is true for $n = k + 1$. Since it is true for $n = 1$, by induction it is true for all positive integers n.

16. Show true for $n = 1$
$4^1 + 6(1) - 1 = 9 = 3(3)$ which is divisible by 3.
Hence the statement is true for $n = 1$

Assume true for $n = k$
$4^k + 6k - 1 = 3m$ where $m \in \mathbb{Z}^+$

Show true for $n = k + 1$
Aim: $4^{k+1} + 6(k + 1) - 1 = 3r$ where $r \in \mathbb{Z}^+$

$4^{k+1} + 6(k + 1) - 1 = 4^k \times 4^1 + 6k + 6 - 1$

$4^{k+1} + 6(k + 1) - 1 = 4^1(4^k) + 6k + 5$

$4^{k+1} + 6(k + 1) - 1 = 4(4^k + 6k - 1 - 6k + 1) + 6k + 5$

$4^{k+1} + 6(k + 1) - 1 = 4(3m - 6k + 1) + 6k + 5$

$4^{k+1} + 6(k + 1) - 1 = 4 \times 3m - 4 \times 6k + 4 \times 1 + 6k + 5$

$4^{k+1} + 6(k + 1) - 1 = 4 \times 3m + 6k(-4 + 1) + (4 + 5)$

$4^{k+1} + 6(k + 1) - 1 = 3 \times 4m + 6k(-3) + 9$

$4^{k+1} + 6(k + 1) - 1 = 3(4m - 6k + 3)$ which is divisible by 6

If the statement is true for $n = k$, then it is true for $n = k + 1$. Since it is true for $n = 1$, by induction it is true for all positive integers n.

17. Show true for $n = 1$
$5^1 + 8(1) + 3 = 16 = 4(4)$ which is divisible by 4.
Hence the statement is true for $n = 1$

Assume true for $n = k$
$5^k + 8k + 3 = 4m$ where $m \in \mathbb{Z}^+$

Show true for $n = k + 1$
Aim: $5^{k+1} + 8(k + 1) + 3 = 4r$ where $r \in \mathbb{Z}^+$

$5^{k+1} + 8(k + 1) + 3 = 5^k \times 5^1 + 8k + 8 + 3$

$5^{k+1} + 8(k + 1) + 3 = 5^1(5^k) + 8k + 11$

$5^{k+1} + 8(k + 1) + 3 = 5(5^k + 8k + 3 - 8k - 3) + 8k + 11$

$5^{k+1} + 8(k + 1) + 3 = 5(4m - 8k - 3) + 8k + 11$

$5^{k+1} + 8(k + 1) + 3 = 5 \times 4m - 5 \times 8k + 5 \times 3 + 8k + 11$

$5^{k+1} + 8(k + 1) + 3 = 4 \times 5m + 8k(-5 + 1) + (-15 + 11)$

$5^{k+1} + 8(k + 1) + 3 = 4 \times 5m + 8k(-4) - 4$

$5^{k+1} + 8(k + 1) + 3 = 4(5m - 8k - 1)$ which is divisible by 4

If the statement is true for $n = k$, then it is true for $n = k + 1$. Since it is true for $n = 1$, by induction it is true for all positive integers n.

18. Show true for $n = 1$
$1^3 - 1 + 3 = 3 = 3(1)$ which is divisible by 3.
Hence the statement is true for $n = 1$

Assume true for $n = k$
$k^3 - k + 3 = 3m$ where $m \in \mathbb{Z}^+$

Show true for $n = k + 1$
Aim: $(k + 1)^3 - (k + 1) + 3 = 3r$ where $r \in \mathbb{Z}^+$

$(k + 1)^3 - (k + 1) + 3 = (k^3 + 3k^2 + 3k + 1) - k - 1 + 3$

$(k + 1)^3 - (k + 1) + 3 = k^3 + 3k^2 + 2k + 3$

$(k + 1)^3 - (k + 1) + 3 = (k^3 - k + 3) + 3k^2 + 3k$

$(k + 1)^3 - (k + 1) + 3 = 3m + 3(k^2 + k)$

$(k + 1)^3 - (k + 1) + 3 = 3(m + k^2 + k)$ which is divisible by 3

If the statement is true for $n = k$, then it is true for $n = k + 1$. Since it is true for $n = 1$, by induction it is true for all positive integers n.

Exercise 26.5D

1. Show true for $n = 1$

LHS	*RHS*
$B^1 = B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

Hence the statement is true when $n = 1$

Assume true for $n = k$

$B^k = \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$

Show true for $n = k + 1$

Aim: $B^{k+1} = \begin{pmatrix} 1 & k + 1 \\ 0 & 1 \end{pmatrix}$

$B^{k+1} = B^k \times B$

$= \begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} 1 & 1 + k \\ 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} 1 & k + 1 \\ 0 & 1 \end{pmatrix}$ as required.

If the statement is true for $n = k$, then it is true for $n = k + 1$. Since it is true for $n = 1$, then by induction it is true for all positive integers n.

2. Show true for $n = 1$

<u>LHS</u>   <u>RHS</u>

$A^1 = A = \begin{pmatrix} 5 & 4 \\ 0 & 1 \end{pmatrix}$ $\begin{pmatrix} 5^1 & 5^1 - 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 4 \\ 0 & 1 \end{pmatrix}$

Hence the statement is true when $n = 1$

Assume true for $n = k$

$A^k = \begin{pmatrix} 5^k & 5^k - 1 \\ 0 & 1 \end{pmatrix}$

Show true for $n = k + 1$

Aim: $A^{k+1} = \begin{pmatrix} 5^{k+1} & 5^{k+1} - 1 \\ 0 & 1 \end{pmatrix}$

$A^{k+1} = A^k \times A$

$= \begin{pmatrix} 5^k & 5^k - 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 5 & 4 \\ 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} 5^k \times 5 & 4 \times 5^k + 5^k - 1 \\ 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} 5^{k+1} & 5 \times 5^k - 1 \\ 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} 5^{k+1} & 5^{k+1} - 1 \\ 0 & 1 \end{pmatrix}$ as required.

If the statement is true for $n = k$, then it is true for $n = k + 1$. Since it is true for $n = 1$, then by induction it is true for all positive integers n.

3. Show true for $n = 1$

<u>LHS</u>   <u>RHS</u>

$C^1 = C = \begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix}$ $\begin{pmatrix} 2^1 & 0 \\ 1 - 2^1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix}$

Hence the statement is true when $n = 1$

Assume true for $n = k$

$C^k = \begin{pmatrix} 2^k & 0 \\ 1 - 2^k & 1 \end{pmatrix}$

Show true for $n = k + 1$

Aim: $C^{k+1} = \begin{pmatrix} 2^{k+1} & 0 \\ 1 - 2^{k+1} & 1 \end{pmatrix}$

$C^{k+1} = C^k \times C$

$= \begin{pmatrix} 2^k & 0 \\ 1 - 2^k & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ -1 & 1 \end{pmatrix}$

$= \begin{pmatrix} 2^k \times 2 & 0 \\ 2 \times (1 - 2^k) - 1 & 1 \end{pmatrix}$

$= \begin{pmatrix} 2^{k+1} & 0 \\ 1 - 2^{k+1} & 1 \end{pmatrix}$ as required.

If the statement is true for $n = k$, then it is true for $n = k + 1$. Since it is true for $n = 1$, then by induction it is true for all positive integers n.

4. Show true for $n = 1$

<u>LHS</u>   <u>RHS</u>

$D^1 = D = \begin{pmatrix} 3 & -2 \\ 0 & 1 \end{pmatrix}$ $\begin{pmatrix} 3^1 & 1 - 3^1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ 0 & 1 \end{pmatrix}$

Hence the statement is true when $n = 1$

Assume true for $n = k$

$D^k = \begin{pmatrix} 3^k & 1 - 3^k \\ 0 & 1 \end{pmatrix}$

Show true for $n = k + 1$

Aim: $D^{k+1} = \begin{pmatrix} 3^{k+1} & 1 - 3^{k+1} \\ 0 & 1 \end{pmatrix}$

$D^{k+1} = D^k \times D$

$= \begin{pmatrix} 3^k & 1 - 3^k \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 3 & -2 \\ 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} 3^k \times 3 & -2 \times 3^k + (1 - 3^k) \\ 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} 3^{k+1} & -3 \times 3^k + 1 \\ 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} 3^{k+1} & 1 - 3^{k+1} \\ 0 & 1 \end{pmatrix}$ as required.

If the statement is true for $n = k$, then it is true for $n = k + 1$. Since it is true for $n = 1$, then by induction it is true for all positive integers n.

5. Show true for $n = 1$

<u>LHS</u>   <u>RHS</u>

$J^1 = J = \begin{pmatrix} 5 & 8 \\ -2 & -3 \end{pmatrix}$ $\begin{pmatrix} 1 + 4(1) & 8(1) \\ -2(1) & 1 - 4(1) \end{pmatrix} = \begin{pmatrix} 5 & 8 \\ -2 & -3 \end{pmatrix}$

Hence the statement is true when $n = 1$

Assume true for $n = k$

$J^k = \begin{pmatrix} 1 + 4k & 8k \\ -2k & 1 - 4k \end{pmatrix}$

Show true for $n = k + 1$

Aim: $J^{k+1} = \begin{pmatrix} 1 + 4(k + 1) & 8(k + 1) \\ -2(k + 1) & 1 - 4(k + 1) \end{pmatrix} =$ $\begin{pmatrix} 4k + 5 & 8k + 8 \\ -2k - 2 & -4k - 3 \end{pmatrix}$

$J^{k+1} = J^k \times J$

$= \begin{pmatrix} 1 + 4k & 8k \\ -2k & 1 - 4k \end{pmatrix}\begin{pmatrix} 5 & 8 \\ -2 & -3 \end{pmatrix}$

$= \begin{pmatrix} (1 + 4k) \times 5 - 2(8k) & 8 \times (1 + 4k) - 3(8k) \\ 5(-2k) - 2(1 - 4k) & 8(-2k) - 3(1 - 4k) \end{pmatrix}$

$= \begin{pmatrix} 4k + 5 & 8k + 8 \\ -2k - 2 & -4k - 3 \end{pmatrix}$ as required.

If the statement is true for $n = k$, then it is true for $n = k + 1$. Since it is true for $n = 1$, then by induction it is true for all positive integers n.

6. Show true for $n = 1$

<u>LHS</u>   <u>RHS</u>

$F^1 = F = \begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix}$ $\begin{pmatrix} 1 & 0 \\ 1 - 2^1 & 2^1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix}$

Hence the statement is true when $n = 1$

Assume true for $n = k$

$F^k = \begin{pmatrix} 1 & 0 \\ 1 - 2^k & 2^k \end{pmatrix}$

Show true for $n = k + 1$

Aim: $F^{k+1} = \begin{pmatrix} 1 & 0 \\ 1 - 2^{k+1} & 2^{k+1} \end{pmatrix}$

$F^{k+1} = F^k \times F$

$= \begin{pmatrix} 1 & 0 \\ 1 - 2^k & 2^k \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -1 & 2 \end{pmatrix}$

$= \begin{pmatrix} 1 & 0 \\ 1 - 2^k - 2^k & 2 \times 2^k \end{pmatrix}$

$= \begin{pmatrix} 1 & 0 \\ 1 - 2 \times 2^k & 2 \times 2^k \end{pmatrix}$

$= \begin{pmatrix} 1 & 0 \\ 1 - 2^{k+1} & 2^{k+1} \end{pmatrix}$ as required.

If the statement is true for $n = k$, then it is true for $n = k + 1$. Since it is true for $n = 1$, then by induction it is true for all positive integers n.

7. Show true for $n = 1$

    LHS _RHS_

    $D^1 = D = \begin{pmatrix} 2 & 2 \\ 0 & 1 \end{pmatrix}$ $\begin{pmatrix} 2^1 & 2(2^1 - 1) \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 0 & 1 \end{pmatrix}$

    Hence the statement is true when $n = 1$

    Assume true for $n = k$

    $D^k = \begin{pmatrix} 2^k & 2(2^k - 1) \\ 0 & 1 \end{pmatrix}$

    Show true for $n = k + 1$

    Aim: $D^{k+1} = \begin{pmatrix} 2^{k+1} & 2(2^{k+1} - 1) \\ 0 & 1 \end{pmatrix}$

    $D^{k+1} = D^k \times D$

    $= \begin{pmatrix} 2^k & 2(2^k - 1) \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 2 \\ 0 & 1 \end{pmatrix}$

    $= \begin{pmatrix} 2^k \times 2 & 2^k \times 2 + 2(2^k - 1) \\ 0 & 1 \end{pmatrix}$

    $= \begin{pmatrix} 2^k \times 2 & 2 \times 2^{k+1} - 2) \\ 0 & 1 \end{pmatrix}$

    $= \begin{pmatrix} 2^{k+1} & 2(2^{k+1} - 1) \\ 0 & 1 \end{pmatrix}$ as required.

    If the statement is true for $n = k$, then it is true for $n = k + 1$. Since it is true for $n = 1$, then by induction it is true for all positive integers n.

8. Show true for $n = 1$

    LHS _RHS_

    $G^1 = G = \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}$ $\begin{pmatrix} 1 & 1 - 2^1 \\ 0 & 2^1 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}$

    Hence the statement is true when $n = 1$

    Assume true for $n = k$

    $G^k = \begin{pmatrix} 1 & 1 - 2^k \\ 0 & 2^k \end{pmatrix}$

    Show true for $n = k + 1$

    Aim: $G^{k+1} = \begin{pmatrix} 1 & 1 - 2^{k+1} \\ 0 & 2^{k+1} \end{pmatrix}$

$G^{k+1} = G^k \times G$

$= \begin{pmatrix} 1 & 1 - 2^k \\ 0 & 2^k \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix}$

$= \begin{pmatrix} 1 & -1 + 2 \times (1 - 2^k) \\ 0 & 2 \times 2^k \end{pmatrix}$

$= \begin{pmatrix} 1 & 1 - 2 \times 2^k \\ 0 & 2^{k+1} \end{pmatrix}$

$= \begin{pmatrix} 1 & 1 - 2^{k+1} \\ 0 & 2^{k+1} \end{pmatrix}$ as required.

If the statement is true for $n = k$, then it is true for $n = k + 1$. Since it is true for $n = 1$, then by induction it is true for all positive integers n.

9. Show true for $n = 1$

    LHS _RHS_

    $E^1 = E = \begin{pmatrix} 3 & 1 \\ 0 & 1 \end{pmatrix}$ $\begin{pmatrix} 3^1 & \frac{1}{2}(3^1 - 1) \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}$

    Hence the statement is true when $n = 1$

    Assume true for $n = k$

    $E^k = \begin{pmatrix} 3^k & \frac{1}{2}(3^k - 1) \\ 0 & 1 \end{pmatrix}$

    Show true for $n = k + 1$

    Aim: $E^{k+1} = \begin{pmatrix} 3^{k+1} & \frac{1}{2}(3^{k+1} - 1) \\ 0 & 1 \end{pmatrix}$

    $E^{k+1} = E^k \times E$

    $= \begin{pmatrix} 3^k & \frac{1}{2}(3^k - 1) \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 3 & 1 \\ 0 & 1 \end{pmatrix}$

    $= \begin{pmatrix} 3^k \times 3 & 3^k + \frac{1}{2}(3^k - 1) \\ 0 & 1 \end{pmatrix}$

    $= \begin{pmatrix} 3^{k+1} & 3^k\left(1 + \frac{1}{2}\right) - \frac{1}{2} \\ 0 & 1 \end{pmatrix}$

    $= \begin{pmatrix} 3^{k+1} & \frac{3}{2}(3^k) - \frac{1}{2} \\ 0 & 1 \end{pmatrix}$

    $= \begin{pmatrix} 3^{k+1} & \frac{1}{2}(3^{k+1} - 1) \\ 0 & 1 \end{pmatrix}$ as required.

    If the statement is true for $n = k$, then it is true for $n = k + 1$. Since it is true for $n = 1$, then by induction it is true for all positive integers n.

10. Show true for $n = 1$

    LHS _RHS_

    $I^1 = I = \begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}$ $\begin{pmatrix} 3^1 & 0 \\ 3(3^1 - 1) & 1 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 6 & 2 \end{pmatrix}$

    Hence the statement is true when $n = 1$

    Assume true for $n = k$

    $I^k = \begin{pmatrix} 3^k & 0 \\ 3(3^k - 1) & 1 \end{pmatrix}$

Show true for $n = k + 1$

Aim: $I^{k+1} = \begin{pmatrix} 3^{k+1} & 0 \\ 3(3^{k+1} - 1) & 1 \end{pmatrix}$

$I^{k+1} = I^k \times I$

$= \begin{pmatrix} 3^k & 0 \\ 3(3^k - 1) & 1 \end{pmatrix} \begin{pmatrix} 3 & 0 \\ 6 & 1 \end{pmatrix}$

$= \begin{pmatrix} 3^k \times 3 & 0 \\ 3 \times 3(3^k - 1) + 6 & 1 \end{pmatrix}$

$= \begin{pmatrix} 3^{k+1} & 0 \\ 3 \times 3(3^k - 3) + 3(2) & 1 \end{pmatrix}$

$= \begin{pmatrix} 3^{k+1} & 0 \\ 3(3^{k+1} - 1) & 1 \end{pmatrix}$ as required.

If the statement is true for $n = k$, then it is true for $n = k + 1$. Since it is true for $n = 1$, then by induction it is true for all positive integers n.

11. Show true for $n = 1$

<u>LHS</u>

$K^1 = K = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

<u>RHS</u>

$\begin{pmatrix} 1 & 1 & \frac{1}{2}(1)(1+1) \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

Hence the statement is true when $n = 1$

Assume true for $n = k$

$K^k = \begin{pmatrix} 1 & k & \frac{1}{2}k(k+1) \\ 0 & 1 & k \\ 0 & 0 & 1 \end{pmatrix}$

Show true for $n = k + 1$

Aim: $K^{k+1} = \begin{pmatrix} 1 & (k+1) & \frac{1}{2}(k+1)((k+1)+1) \\ 0 & 1 & (k+1) \\ 0 & 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} 1 & (k+1) & \frac{1}{2}(k+1)(k+2) \\ 0 & 1 & (k+1) \\ 0 & 0 & 1 \end{pmatrix}$

$K^{k+1} = K^k \times K$

$= \begin{pmatrix} 1 & k & \frac{1}{2}k(k+1) \\ 0 & 1 & k \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} 1 & (k+1) & \frac{1}{2}(k+1) + k + 1 \\ 0 & 1 & k \\ 0 & 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} 1 & (k+1) & \frac{1}{2}[k(k+1) + 2k + 2] \\ 0 & 1 & k \\ 0 & 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} 1 & (k+1) & \frac{1}{2}[k^2 + 3k + 2] \\ 0 & 1 & k \\ 0 & 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} 1 & (k+1) & \frac{1}{2}(k+1)(k+2) \\ 0 & 1 & (k+1) \\ 0 & 0 & 1 \end{pmatrix}$ as required.

If the statement is true for $n = k$, then it is true for $n = k + 1$. Since it is true for $n = 1$, then by induction it is true for all positive integers n.

12. Show true for $n = 1$

<u>LHS</u>

$L^1 = L = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

<u>RHS</u>

$\begin{pmatrix} 1 & 1 & \frac{1}{2}(1)^2 + 3(1)) \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

Hence the statement is true when $n = 1$

Assume true for $n = k$

$L^k = \begin{pmatrix} 1 & k & \frac{1}{2}(k^2 + 3k) \\ 0 & 1 & k \\ 0 & 0 & 1 \end{pmatrix}$

Show true for $n = k + 1$

Aim: $L^{k+1} = \begin{pmatrix} 1 & (k+1) & \frac{1}{2}((k+1)^2 + 3(k+1)) \\ 0 & 1 & (k+1) \\ 0 & 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} 1 & (k+1) & \frac{1}{2}(k^2 + 5k + 4) \\ 0 & 1 & (k+1) \\ 0 & 0 & 1 \end{pmatrix}$

$L^{k+1} = L^k \times L$

$= \begin{pmatrix} 1 & k & \frac{1}{2}(k^2 + 3k) \\ 0 & 1 & k \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} 1 & (k+1) & \frac{1}{2}(k+1) + k + 1 \\ 0 & 1 & k \\ 0 & 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} 1 & (k+1) & \frac{1}{2}[k(k+1) + 2k + 2] \\ 0 & 1 & k \\ 0 & 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} 1 & (k+1) & \frac{1}{2}[k^2 + 3k + 2] \\ 0 & 1 & k \\ 0 & 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} 1 & (k+1) & \frac{1}{2}(k^2 + 5k + 4) \\ 0 & 1 & (k+1) \\ 0 & 0 & 1 \end{pmatrix}$ as required.

If the statement is true for $n = k$, then it is true for $n = k + 1$. Since it is true for $n = 1$, then by induction it is true for all positive integers n.

Exercise 26.5E

1. $n = 1$

LHS	RHS
$3^1 = 3$	$2(1) + 1 = 3$

$3 = 3$ so statement is true for $n = 1$

Assume true for $n = k$

$3^k \geq 2k$

Show true for $n = k + 1$

Aim: $3^{k+1} \geq 2(k + 1)$

$3^{k+1} = 3 \times 3^k$

$3 \times 3^k > 3 \times (2k)$

$3 \times 3^k > 6k > 2(k + 1)$

If the statement is true for $n = k$, then it is true for $n = k + 1$. Since it is true for $n = 1$, by induction it is true for all positive integers n.

2. $n = 7$

LHS	RHS
$7! = 5040$	$3^7 = 2187$

$5040 > 2187$ so statement is true for $n = 7$

Assume true for $n = k$

$k! \geq 3^k$

Show true for $n = k + 1$

Prove $(k + 1)! > 3^{k+1}$

$(k + 1)! = (k + 1) \times k!$

$(k + 1) \times k! > (k + 1) \times 3^k$

Since $n \geq 7$, $(k + 1) \times 3^k > 3^{k+1}$

$(k + 1)! > 3^{k+1}$

If the statement is true for $n = k$, then it is true for $n = k + 1$. Since it is true for $n = 1$, by induction it is true for all positive integers n.

3. $n = 3$

LHS	RHS
$2^3 = 8$	$2(3) = 6$

$8 > 6$ so statement is true for $n = 3$

Assume true for $n = k$

$2^k > 2k$

Show true for $n = k + 1$

Aim: $2^{k+1} > 2(k + 1)$

$2^{k+1} = 2 \times 2^k$

$2 \times 2^k > 2 \times (2k)$

$2 \times 2^k > 4k$

Since $n \geq 7$, $4k > 2(k + 1)$

$2 \times 2^k > 2(k + 1)$

If the statement is true for $n = k$, then it is true for $n = k + 1$. Since it is true for $n = 1$, by induction it is true for all positive integers n.

4. $n = 5$

LHS	RHS
$2^5 = 32$	$5^2 = 25$

$32 > 25$ so statement is true for $n = 5$

Assume true for $n = k$

$2^k > k^2$

Show true for $n = k + 1$

Aim: $2^{k+1} > (k + 1)^2$

$2^{k+1} = 2 \times 2^k$

$2 \times 2^k > 2 \times k^2$

Since $n \geq 5$, $2k^2 > (k + 1)^2$

$2 \times 2^k > (k + 1)^2$

If the statement is true for $n = k$, then it is true for $n = k + 1$. Since it is true for $n = 1$, by induction it is true for all positive integers n.

5. $n = 4$

LHS	RHS
$3^4 = 81$	$(4 + 1)! = 120$

$81 < 120$ so statement is true for $n = 5$

Assume true for $n = k$

$3^k < (k + 1)!$

Show true for $n = k + 1$

Aim: $3^{k+1} < (k + 1 + 1)! = (k + 2)!$

$3^{k+1} = 3 \times 3^k$

$3 \times 3^k < 3 \times (k + 1)!$

Since $n \geq 4$, $(k + 1)! < (k + 2)!$

$3 \times 3^k < (k + 2)^2$

If the statement is true for $n = k$, then it is true for $n = k + 1$. Since it is true for $n = 1$, by induction it is true for all positive integers n.

6. $n = 7$

LHS	RHS
$6! = 720$	$6^3 = 216$

$720 > 216$ so statement is true for $n = 6$

Assume true for $n = k$

$k! > k^3$

Show true for $n = k + 1$

Prove $(k + 1)! > (k + 1)^3$

$(k + 1)! = (k + 1) \times k!$

$(k + 1) \times k! > (k + 1) \times k^3$

Since $n \geq 6$, $(k + 1) \times k^3 > (k + 1)^3$

$(k + 1)! > (k + 1)^3$

If the statement is true for $n = k$, then it is true for $n = k + 1$. Since it is true for $n = 1$, by induction it is true for all positive integers n.

7. $n = 1$

<u>LHS</u>   <u>RHS</u>

$1! = 1$ $1^1 = 1$

$1 = 1$ so statement is true for $n = 1$

Assume true for $n = k$

$k! \leq k^k$

Show true for $n = k + 1$

Prove $(k + 1)! \leq (k + 1)^{(k + 1)}$

$(k + 1)! = (k + 1) \times k!$

$(k + 1) \times k! \leq (k + 1) \times k^k$

Since $n \geq 1$, $(k + 1) \times k^k \leq (k + 1)^{(k + 1)}$

$(k + 1)! > (k + 1)^{(k + 1)}$

If the statement is true for $n = k$, then it is true for $n = k + 1$. Since it is true for $n = 1$, by induction it is true for all positive integers n.

8. $n = 1$

<u>LHS</u>   <u>RHS</u>

$6(6) + 6 = 42$ $2^6 = 64$

$42 < 64$ so statement is true for $n = 6$

Assume true for $n = k$

$6k + 6 < 2^k$

Show true for $n = k + 1$

Prove $6(k + 1) + 6 < 2^{(k + 1)}$

$6(k + 1) + 6 = (6k + 6) + 6$

$(6k + 6) + 6 < 2^k + 6$

Since $n \geq 6$, $2^k + 6 < 2^{(k + 1)}$

$6(k + 1) < 2^{(k + 1)}$

If the statement is true for $n = k$, then it is true for $n = k + 1$. Since it is true for $n = 1$, by induction it is true for all positive integers n.

9. $n = 1$

<u>LHS</u>   <u>RHS</u>

$\sum_{r=1}^{n} 1 = 1$ $\frac{1}{2}1^2 = \frac{1}{2}$

$1 > \frac{1}{2}$ so statement is true for $n = 1$

Assume true for $n = k$

$\sum_{r=1}^{k} r > \frac{1}{2}k^2$

Show true for $n = k + 1$

Prove $\sum_{r=1}^{k+1} r > \frac{1}{2}(k + 1)^2$

$\sum_{r=1}^{k+1} r = \sum_{r=1}^{k} r + (k + 1)$

$\sum_{r=1}^{k+1} r > \frac{1}{2}k^2 + (k + 1)$

Since $n \geq 1$, $\frac{1}{2}k^2 + (k + 1) > \frac{1}{2}(k^2 + 2k + 1)$

$\sum_{r=1}^{k+1} r > \frac{1}{2}(k + 1)^2$

If the statement is true for $n = k$, then it is true for $n = k + 1$. Since it is true for $n = 1$, by induction it is true for all positive integers n.

10. $n = 1$

<u>LHS</u>   <u>RHS</u>

$\sum_{r=1}^{n} 1^2 = 1$ $\frac{1}{4}(1)(1 + 1)^2 = 1$

$1 = 1$ so statement is true for $n = 1$

Assume true for $n = k$

$\sum_{r=1}^{k} r^2 \geq \frac{1}{4}k(k + 1)^2$

Show true for $n = k + 1$

Prove $\sum_{r=1}^{k+1} r > \frac{1}{4}(k + 1)^2(k + 1 + 1)^2$

$\sum_{r=1}^{k+1} r^2 = \sum_{r=1}^{k} r^2 + (k + 1)^2$

$\sum_{r=1}^{k+1} r^2 \geq \frac{1}{4}k(k + 1)^2 + (k + 1)^2$

Since $n \geq 1$, $\frac{1}{4}k(k + 1)^2 + (k + 1)^2 > \frac{1}{4}(k + 1)(k + 2)^2$

$\sum_{r=1}^{k+1} r^2 \geq \frac{1}{4}(k + 1)(k + 2)^2$

If the statement is true for $n = k$, then it is true for

$n = k + 1$. Since it is true for $n = 1$, by induction it is true for all positive integers n.

Exercise 26.5F

1. $n = 1$

<u>LHS</u>   <u>RHS</u>

$\frac{d}{dx} = 4e^{4x}$ $4^1 e^{4x} = 4e^{4x}$

Result is true for $n = 1$

Assume true for $n = k$

$$\frac{d^k}{dx^k}e^{4x} = 4^k e^{4x}$$

Show true for $n = k + 1$

Prove $\dfrac{d^{k+1}}{dx^{k+1}}e^{4x} = 4^{k+1}e^{4x}$

$$\frac{d^{k+1}}{dx^{k+1}}e^{4x} = \frac{d}{dx}\left(\frac{d^k}{dx^k}e^{4x}\right)$$

$$= \frac{d}{dx}(4^k e^{4x})$$

$$= 4 \times 4^k e^{4x}$$

$$= 4^{k+1}e^{4x} \text{ as required}$$

If the statement is true for $n = k$, then it is true for $n = k + 1$. Since it is true for $n = 1$, by induction it is true for all positive integers n.

2. $n = 1$

LHS   _RHS_

$\dfrac{d}{dx} = e^x + xe^x$ $(1 + x)e^x = e^x + xe^x$

Result is true for $n = 1$

Assume true for $n = k$

$$\frac{d^k}{dx^k}xe^x = (k + x)e^x = ke^x + xe^x$$

Show true for $n = k + 1$

Prove $\dfrac{d^{k+1}}{dx^{k+1}}xe^x = (k + 1 + x)e^x$

$$\frac{d^{k+1}}{dx^{k+1}}xe^x = \frac{d}{dx}\left(\frac{d^k}{dx^k}xe^x\right)$$

$$= \frac{d}{dx}(ke^x + xe^x)$$

$$= ke^x + e^x + xe^x$$

$$= (k + 1 + x)e^x \text{ as required}$$

If the statement is true for $n = k$, then it is true for $n = k + 1$. Since it is true for $n = 1$, by induction it is true for all positive integers n.

3. $n = 1$

LHS

$(\cos\theta + i\sin\theta)^1 = (\cos\theta + i\sin\theta)$

RHS

$(\cos(1)\theta + i\sin(1)\theta) = (\cos\theta + i\sin\theta)$

Result is true for $n = 1$

Assume true for $n = k$

$(\cos\theta + i\sin\theta)^k = (\cos k\theta + i\sin k\theta)$

Show true for $n = k + 1$

Prove $(\cos\theta + i\sin\theta)^{k+1} = (\cos(k+1)\theta + i\sin(k+1)\theta)$

$(\cos\theta + i\sin\theta)^{k+1} = (\cos\theta + i\sin\theta)^k \times (\cos\theta + i\sin\theta)$

$= (\cos k\theta + i\sin k\theta) \times (\cos\theta + i\sin\theta)$

$= \cos k\theta \times \cos\theta + \cos k\theta \times i\sin\theta + i\sin k\theta \times \cos\theta + i\sin$

$k\theta \times i\sin\theta$

$= (\cos k\theta \times \cos\theta - \sin k\theta \times \sin\theta) + i(\cos k\theta \times \sin\theta + \sin k\theta \times \cos\theta)$

$= \cos(k+1)\theta + i\sin(k+1)\theta$ as required

If the statement is true for $n = k$, then it is true for $n = k + 1$. Since it is true for $n = 1$, by induction it is true for all positive integers n.

4. $n = 1$

LHS

$\dfrac{d}{dx} = (xy + 3y = 1) \rightarrow y + x\dfrac{dy}{dx} + 3\dfrac{dy}{dx} = 0 \rightarrow \dfrac{dy}{dx}(x + 3) + y = 0$

RHS

$(x + 3)\dfrac{dy}{dx} + (1)\dfrac{d^0 y}{dx^0} = 0 \rightarrow \dfrac{dy}{dx}(x + 3) + y = 0$

Result is true for $n = 1$

Assume true for $n = k$

$$\frac{d^k}{dx^k}(xy + 3y = 1) = (x + 3)\frac{d^k y}{dx^k} + k\frac{d^{k-1}y}{dx^{k-1}} = 0$$

Show true for $n = k + 1$

Prove $\dfrac{d^{k+1}}{dx^{k+1}}(xy + 3y = 1) = (x + 3)\dfrac{d^{k+1}y}{dx^{k+1}} + (k + 1)\dfrac{d^k y}{dx^k} = 0$

$$\frac{d^{k+1}}{dx^{k+1}}(xy + 3y = 1) = \frac{d}{dx}\left(\frac{d^k}{dx^k}(xy + 3y = 1)\right)$$

$$= \frac{d}{dx}\left((x + 3)\frac{d^k y}{dx^k} + k\frac{d^{k-1}y}{dx^{k-1}} = 0\right)$$

$$= \frac{d^k y}{dx^k} + (x + 3)\frac{d^{k+1}y}{dx^{k+1}} + k\frac{d^k y}{dx^k} = 0$$

$$= (x + 3)\frac{d^{k+1}y}{dx^{k+1}} + (k + 1)\frac{d^k y}{dx^k} = 0 \text{ as required}$$

If the statement is true for $n = k$, then it is true for $n = k + 1$. Since it is true for $n = 1$, by induction it is true for all positive integers n.

5. $n = 2$

LHS   _RHS_

$\dfrac{d^2}{dx^2}(\sin 3x) = -9\sin 3x$ $(-1)^{\frac{2}{2}} \times 3^2 \times \sin 3x = -9\sin 3x$

Result is true for $n = 2$

Assume true for $n = k$

$$\frac{d^k}{dx^k}\sin 3x = (-1)^{\frac{k}{2}} \times 3^k \times \sin 3x$$

Show true for $n = k + 2$

Prove $\dfrac{d^{k+2}y}{dx^{k+2}}\sin 3x = (-1)^{\frac{k+2}{2}} \times 3^{k+2} \times \sin 3x$

$$\frac{d^{k+1}y}{dx^{k+1}}\sin 3x = \frac{d}{dx}\left(\frac{d^k}{dx^k}\sin 3x\right)$$

$$= \frac{d}{dx}\left((-1)^{\frac{k}{2}} \times 3^k \times \sin 3x\right)$$

$$= (-1)^{\frac{k}{2}} \times 3^k \times \cos 3x \times 3$$

$$= (-1)^{\frac{k}{2}} \times 3^{k+1} \times \cos 3x$$

$$\frac{d^{k+2}y}{dx^{k+2}} \sin 3x = \frac{d}{dx}\left((-1)^{\frac{k}{2}} \times 3^{k+1} \times \cos 3x\right)$$

$$= (-1)^{\frac{k}{2}} \times 3^{k+1} \times \sin 3x \times 3 \times (-1)$$

$$= (-1)^{\frac{k+2}{2}} \times 3^{k+2} \times \sin 3x \text{ as required}$$

If the statement is true for $n = k$, then it is true for $n = k + 1$. Since it is true for $n = 1$, by induction it is true for all positive integers n.

Exercise 26.6

26.1

(a) $n = 2k$

(b) $n = 2k + 1$

Then $7n + 4 = 7(2k) + 4$

Then $8n - 2 = 8(2k + 1) - 2$

$= 2(7k + 2)$ which is even.

$= 16k + 6$

$= 2(8k + 3)$ which is even.

26.2 $x = 5$ is a counterexample

26.3 Assume x is irrational, then $x + 2$ is rational

Then $x + 2 = \frac{a}{b}$, where a and b have no common factors.

$x = \frac{a}{b} - 2$

$= \frac{a - 2b}{b}$ which is rational since if $a, b \in \mathbb{Z}$ then $a - 2b \in \mathbb{Z}$.

This contradicts the assumption that x is irrational. Our assumption must be false and the original statement is proven true.

26.4

(a) Contrapositive: 'If x is even, then $7x - 5$ is even'.

$x = 2k$

$7x - 5 = 7(2k) - 5$

$= 14k - 5$

$= 2(7k - 3) + 1$ which is odd.

Contrapositive is true therefore original statement is true.

(b) Contrapositive: 'If x is odd, then $x^2 - 3x + 5$ is odd'

$n = 2k + 1$

$x^2 - 3x + 5 = (2k + 1)^2 - 3(2k + 1) + 5$

$= 4k^2 - 2k + 3$

$= 2(2k^2 - k + 1) + 1$ which is odd.

Contrapositive is true therefore original statement is true.

26.5A Show true for $n = 1$

$\underline{LHS}$ $\underline{RHS}$

$A^1 = A = \begin{pmatrix} -2 & -1 \\ 9 & 4 \end{pmatrix}$ $\begin{pmatrix} 1 - 3(1) & -1 \\ 9(1) & 3(1) + 1 \end{pmatrix} = \begin{pmatrix} -2 & -1 \\ 9 & 4 \end{pmatrix}$

Hence the statement is true when $n = 1$

Assume true for $n = k$

$A^k = \begin{pmatrix} 1 - 3k & -k \\ 9k & 3k + 1 \end{pmatrix}$

Show true for $n = k + 1$

Aim: $A^{k+1} = \begin{pmatrix} 1 - 3(k + 1) & -(k + 1) \\ 9(k + 1) & 3(k + 1) + 1 \end{pmatrix}$

$= \begin{pmatrix} -2 - 3k & -(k + 1) \\ 9k + 9 & 3k + 4 \end{pmatrix}$

$A^{k+1} = A^k \times A$

$= \begin{pmatrix} 1 - 3k & -k \\ 9k & 3k + 1 \end{pmatrix}\begin{pmatrix} -2 & -1 \\ 9 & 4 \end{pmatrix}$

$= \begin{pmatrix} -2 \times (1 - 3k) + 9 \times (-k) & -1 \times (1 - 3k) + 4 \times (-k) \\ -2 \times 9k + 9 \times (3k + 1) & -1 \times -9k + 4 \times (3k + 1) \end{pmatrix}$

$= \begin{pmatrix} -2 - 3k & -1 - k \\ 9k + 9 & 3k + 4 \end{pmatrix}$

$= \begin{pmatrix} -2 - 3k & -(k + 1) \\ 9k + 9 & 3k + 4 \end{pmatrix}$ as required.

If the statement is true for $n = k$, then it is true for $n = k + 1$. Since it is true for $n = 1$, then by induction it is true for all positive integers n.

26.5B Show true for $n = 1$

$LHS = \sum_{r=1}^{1} 1(1 + 4) = 5$

$RHS = \frac{1}{6}(1)(1 + 1)(2(1) + 13) = 5$

So result is true when $n = 1$

Assume true for $n = k$

$\sum_{r=1}^{k} r(r + 4) = \frac{1}{6}k(k + 1)(2k + 13)$

Show true for $n = k + 1$

Aim: $\sum_{r=1}^{k+1} r(r + 4) = \frac{1}{6}(k + 1)((k + 1) + 1)(2(k + 1) + 13) = \frac{1}{6}(k + 1)(k + 2)(2k + 15)$

$\sum_{r=1}^{k+1} r(r + 4) = \sum_{r=1}^{k} r(r + 4) + (k + 1)((k + 1) + 4)$

$= \frac{1}{6}k(k + 1)(2k + 13) + (k + 1)(k + 5)$

$= \frac{1}{6}(k + 1)[k(2k + 13) + 6(k + 5)]$

$= \frac{1}{6}(k + 1)[2k^2 + 19k + 30]$

$= \frac{1}{6}(k + 1)(k + 2)(2k + 15)$ as required.

If the statement is true for $n = k$, then it is true for $n = k + 1$. Since it is true for $n = 1$, by induction it is true for all positive integers n.